

Flows and Maps: A Geometric invitation to Nonlinear Analysis

Kevin R. Vixie

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Contents

1	Normal Maps and Flows	5
2	Two Intertwined Stories	6
2.1	The Normal Flow of 1-dimensional Sets	7
2.2	Normal flow N_ϵ in more than one dimension	9
2.3	Gauss Map G for 1-dimensional manifolds	10
2.4	Gauss Map for higher dimensional manifolds	11
2.5	Up Next	13
3	The Derivative Path	14
3.1	Finding the Right Definition	14
3.2	Understanding the Right Definition	14
3.3	The $D_x f$ vs. $\det(D_x f)$	15
3.4	Plumbing the Depths of $\int_a^b f' = f(b) - f(a)$	18
3.5	Arriving at Two Versions of the Area formula	18
3.6	Next Stories to Follow	20
4	Degree Theory	21
4.1	A Motivating Exercise	21
4.2	Topologies and Equivalence	21
4.3	Degree Theory in 1 Dimension	23
4.3.1	Sard's Theorem for Maps from $\mathbb{R}$ to $\mathbb{R}$	24
4.3.2	The Sum in question is finite for $y \in \mathcal{R}_f$	27
4.3.3	Homotopies that do not move $\partial[a, b]$ leave the degree invariant	28
4.4	Degree Theory in Higher Dimensions	28
4.4.1	Sard's Theorem for Maps from $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n, n > 1$	29
4.4.2	Finite Sums, Again	33
4.4.3	Homotopy Invariance of Degree in Higher Dimensions	33
4.4.4	Orientation and Mappings in Co-Dimension 1	35
4.4.5	Manifolds: Degree for maps between manifolds.	37
4.5	References on Degree Theory	39
4.6	Next Steps	39
5	The Co-area Formula: Lipschitz Maps, Rectifiability, and Calculus in the Wild	41
5.1	Review of the Area Formula for $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$	43
5.2	Extension of the Area Formula to $f : \mathbb{R}^n \rightarrow \mathbb{R}^m, n < m$	44
5.3	The Co-area Formula: the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m, n > m$	45
5.4	Lipschitz Maps	47
5.5	Rectifiable Sets	51
5.6	The Area and Co-area Formulas in the nonsmooth case	54
5.7	Examples and Notes on Calculus in the Wild	55
5.7.1	Two Examples	55
5.7.2	Almost Sard's Theorem	58
5.7.3	Level sets of the Distance Function	59
5.7.4	A Rectifiable Measure	59
5.7.5	Local Approximation and Densities	62

5.7.6	Many Different Boundaries	62
6	Integrating the Gaussian Curvature: Gauss-Bonnet	64
6.1	Smoothly Homotopic $n - 1$ -Submanifolds	64
6.2	Compact 2-dimensional Submanifolds of $\mathbb{R}^3$	67
7	Evolving the Volume: Tube Formulas and Reach	70
7.1	Positive Reach	70
7.2	The Tube Formula for (Smooth) Normal Flow	72
7.3	Generalization to Convex Bodies	73
7.3.1	The Square	77
7.3.2	The Cube	78
8	Comments and References for Further Study	80
9	Acknowledgments	82

1 Normal Maps and Flows

The rich world opened by the action of maps and flows, on sets and measures in $\mathbb{R}^n$, can be deeply illuminated by the careful study of two simple examples – flow of a set along the set of vectors normal to a smooth, simple closed curve in $\mathbb{R}^2$, and the Gauss map, in which we map each point on that curve to the unit normal to the curve at that point. See Figure 1.

In these notes I trace the story that unfolds through that study, following the generalizations to higher dimensions and the tools that naturally emerge as important in the quest to understand. In the course of this study, we will encounter derivatives, degree theory, mean curvature, Gaussian curvature, tube formulas, curvature measures, Sard’s theorem, as well as the notions of positive reach and regular level sets.

While the natural structure of the guiding stories (illustrated very simply in Figure 1), the central technical tools, and the useful generalizations, is best organized as a graph or tree with loops (rather than a linear progression), the text is of course written sequentially. In the next section, I give a less linear summary of what we will encounter in our explorations, starting with the simplest examples in order to indicate how things are woven together and what relates to what. Hopefully, this overview will enable you, the reader, to know where things go as they are developed in the following sections.

Remark 1.1. *I will often use the Hausdorff measures $\mathcal{H}^k$ even when the Lebesgue measures $\mathcal{L}^k$ make sense, because, when Lebesgue measure makes sense, the Hausdorff measure and the Lebesgue measure are identical.*

Remark 1.2. *These notes are intended to spark an interest in further study of the topics exposed, to help the reader begin seeing the connections between pieces not always explained in such a way as to make the connections clear. I have written an informal exposition that will undoubtedly make the more precise souls out there uneasy. The solution for them is easy – look at the references I highlight in the last section of the notes and the bibliography. They are all less informal.*

2 Two Intertwined Stories

We begin our exploration with smooth, closed, non-intersecting curves in $\mathbb{R}^2$. These curves, which can be seen to be topologically equivalent to (i.e. continuously equivalent to) circles, will allow us to explore a small universe of ideas that generalize very naturally to $n-1$ -dimensional surfaces in $\mathbb{R}^n$.

Now for the curves. Figure 1 shows a curve that is then mapped with one of two mappings - a normal flow, in which it is pushed out a distance ϵ along the normal vectors to the curve and the Gauss map, which maps each point p of the curve to the outward pointing unit normal vector to the curve at p .

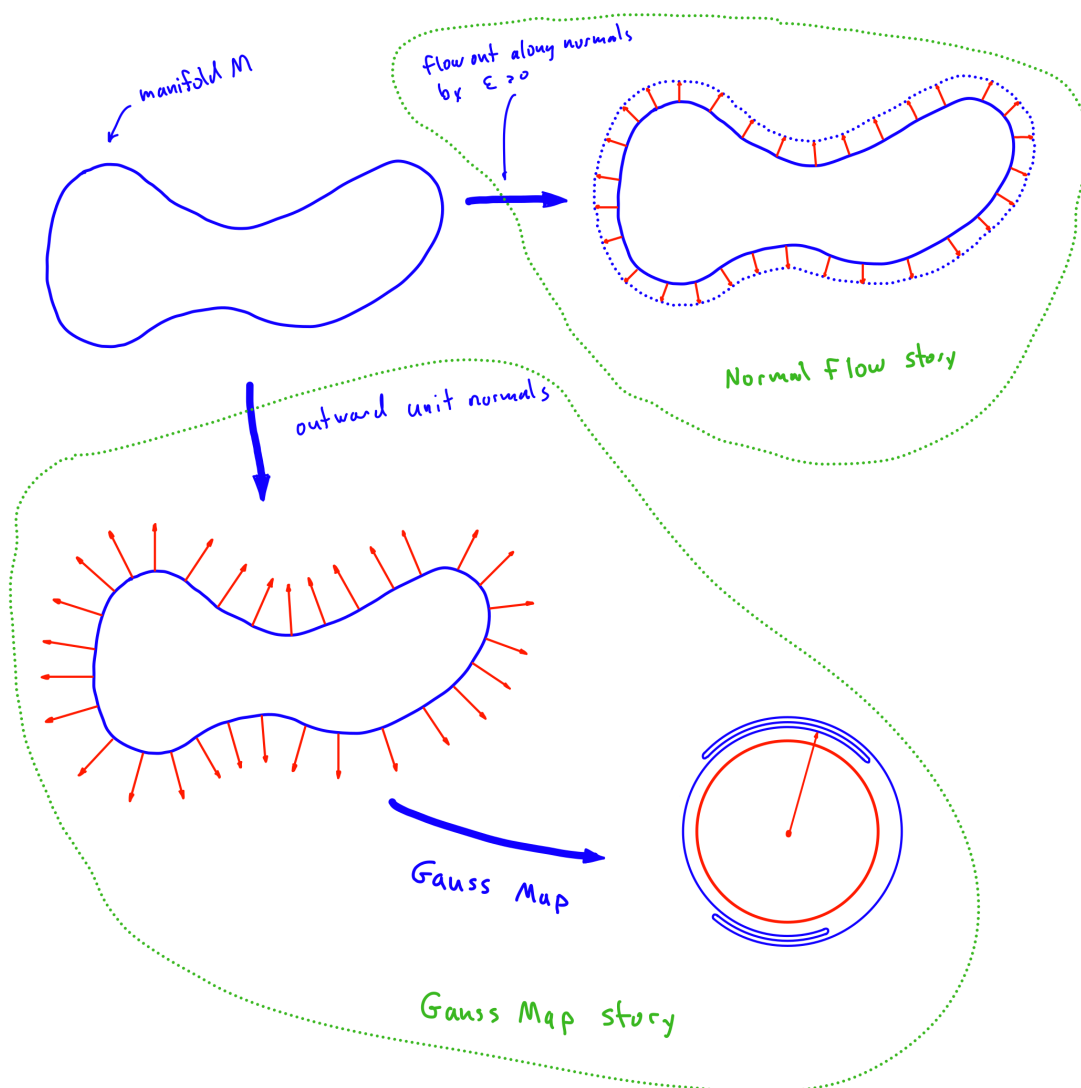


Figure 1: Normal flows and the Gauss map in co-dimension 1.

A careful study of these two maps opens a rich set of ideas and tools for nonlinear analysis of and on $n-1$ dimensional submanifolds (surfaces) in $\mathbb{R}^n$. Many of these concepts and tools, are central to the study of k -dimensional “rough” surfaces called rectifiable sets in $\mathbb{R}^n$.

2.1 The Normal Flow of 1-dimensional Sets

Coming back to the closed, non-intersecting curve M (a 1-dimensional manifold) and to the first story of normal flow, a natural first question to ask is how the length of the curve changed when we pushed it out a distance ϵ . To answer this, we first observe that for a smooth (at least twice differentiable) curve we know that we can find, at each point p on the curve, a best fitting circle that not only is tangent to the circle at p but also matches the second order derivative of the curve.

Let $\gamma : [0, L] \rightarrow M \subset \mathbb{R}^2$ be our unit speed ($|\dot{\gamma}(t)| = 1$ for all $t \in [0, L]$), counterclockwise parameterization of M (outside is to the right of the forward direction in the progression around the curve) and notice that in this case, $\kappa \equiv |\ddot{\gamma}(t)|$ (which we name the *curvature*) is precisely the rate at which the tangent vector turns (radians per unit distance) as we move along the curve.

For any unit speed tangent vector $\gamma(t)$ we have

$$\frac{d \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}{dt} = 0$$

implies

$$\langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle = 0.$$

For a circle of radius r , a little bit of thought and a sketch or two should be sufficient to convince you that $\kappa = \frac{2\pi}{2\pi r} = \frac{1}{r}$ and explains why the center of the osculating circle (circle matching the curve up to second order) is a distance $\frac{1}{\kappa}$ along the normal in the *inward* direction.

Now we notice that if we push a piece of a circle of radius r out along the normals a distance ϵ , its length changes from Δl to $(\frac{r+\epsilon}{r})\Delta l = (1 + \epsilon\kappa)\Delta l$. Wherever the osculating circle has its center also along the outward normal, the curvature is negative since the tangent vector is turning in the clockwise (as opposed to the counterclockwise) direction at that point. Accordingly if we push the curve a distance ϵ towards the center of the osculating circle (which is now in the outward direction), Δl becomes $(1 - \epsilon\frac{1}{r})\Delta l = (1 + \epsilon\kappa)\Delta l$. See Figures 2 and 3

Using these factors, we can now easily find the length of the curve after it has been pushed out along the normal directions a distance ϵ . Using $N_d(\gamma)$ to denote the piece of curve pushed out a distance d , we get that

$$\mathcal{H}^1(N_\epsilon(\gamma)) = \int_{\gamma} (1 + \epsilon\kappa(s)) d\mathcal{H}^1 s$$

See Figure 4.

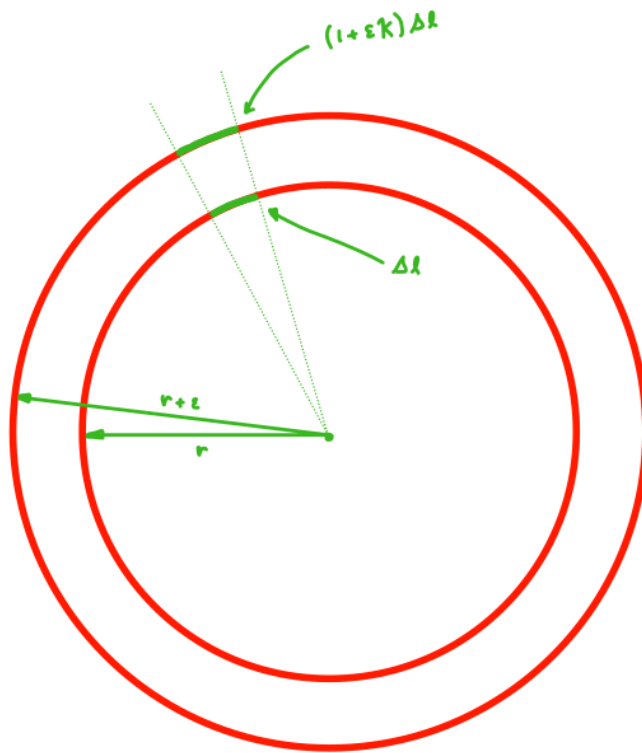


Figure 2: Curvature and the length changes under normal flows

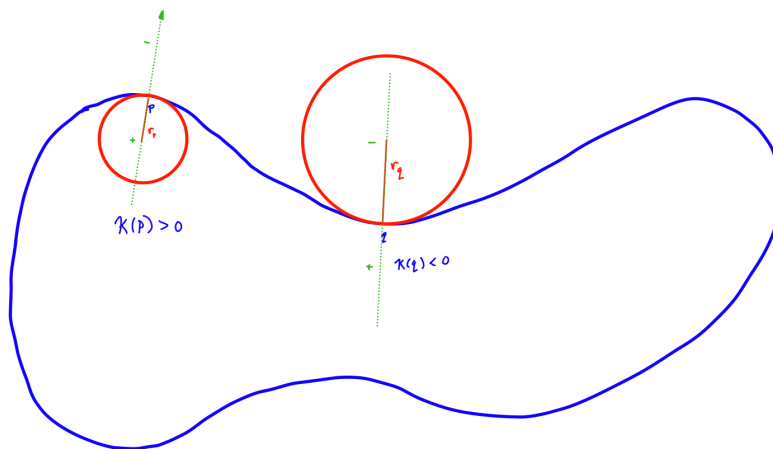


Figure 3: Osculating circles, and positive and negative curvatures

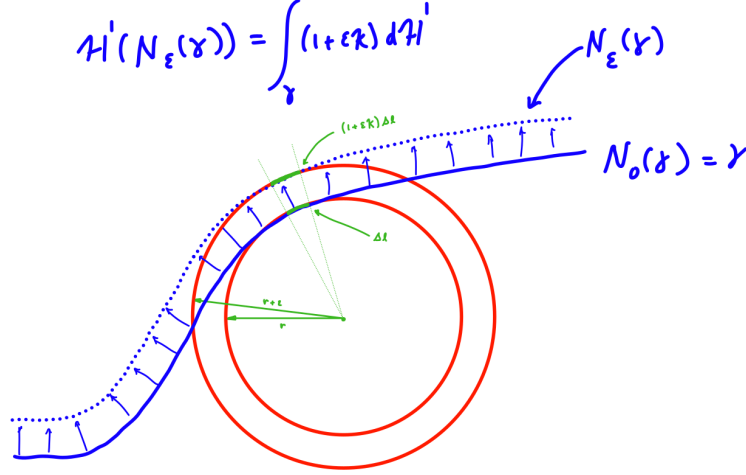


Figure 4: Calculation of changes in length using curvatures.

In summary, we get that the length of the curve M , pushed out a distance ϵ along the normal lines, is given by a polynomial in ϵ

$$\begin{aligned}\mathcal{H}^1(N_\epsilon(M)) &= \int_\gamma 1 \, d\mathcal{H}^1 s + \epsilon \int_M \kappa(s) \, d\mathcal{H}^1 s \\ &= \mathcal{H}^1(M) + \epsilon 2\pi\end{aligned}$$

but the last integral is always 2π for simple closed curves since it is just a measure of how much we turned and it is clear that we have made one full rotation (plus, perhaps, a bunch of wild back and forths) as we go around the circle.

Remark 2.1 (Positive reach). *We assume M , the 1-dimensional set we are flowing out, has positive reach. At each point of M , we have an outward directed normal line. Normal lines may intersect. If there is a distance $R_M > 0$ such that if we are closer to M than R_M , there is no intersection of normal lines, we say that the reach of M is at least R_M . The important implication is that the flow along the normal lines is well behaved up until an intersection. So, when the set has positive outward reach, we know that the calculation outlined above is valid.*

2.2 Normal flow N_ϵ in more than one dimension

At any point p on an $n-1$ dimensional surface in $\mathbb{R}^n$, we may choose orthogonal coordinates $\{e_i\}_{i=1}^n$ for $\mathbb{R}^n$ so that $p = 0$, the surface satisfies

$$x_n = \sum_{i=1}^{n-1} a_i x_i^2,$$

and the outward normal vector is $(0, 0, \dots, 0, 1)$.

With this choice of coordinates, we get that this implies that the curvatures in each of the $n - 1$ directions are given by $\kappa_i = -2a_i$ for $i = 1, \dots, n - 1$ so that

$$x_n = \sum_{i=1}^{n-1} -\frac{\kappa_i}{2} x_i^2.$$

The important thing to note (either by careful geometric observation or calculation) is that the derivative of the normal vector field in the directions of $\{e_i\}_{i=1}^{n-1}$ has the property that

$$\left\langle \frac{dN}{de_i}, e_j \right\rangle = \kappa_i \delta_i^j \text{ or, equivalently } \frac{dN}{de_i} = \kappa_i e_i$$

It also means that the derivative of the normal map, which sends a point to the the point plus ϵ times the normal

$$N_\epsilon : x \rightarrow x + \epsilon N_x,$$

is given by

$$\begin{aligned} D_x N_\epsilon &= I_{n-1} + \epsilon \operatorname{diag}(\kappa_1, \kappa_2, \dots, \kappa_{n-1}) \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \epsilon \begin{bmatrix} \kappa_1 & 0 & \cdots & 0 \\ 0 & \kappa_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \kappa_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \epsilon \kappa_1 & 0 & \cdots & 0 \\ 0 & 1 + \epsilon \kappa_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 + \epsilon \kappa_{n-1} \end{bmatrix} \end{aligned}$$

We conclude that the determinant of the derivative of N_ϵ is given by

$$\det(DN_\epsilon) = \prod_{i=1}^{n-1} (1 + \epsilon \kappa_i)$$

See Figure 5 for the case in which $n = 3$.

2.3 Gauss Map G for 1-dimensional manifolds

The Gauss map sends a point on a co-dimension 1 submanifold of $\mathbb{R}^2$ to the outward normal vector at that point (here we are thinking of simple closed curves dividing the plane into two regions, one bounded, the other - the outside - unbounded).

In one dimension the determinant of that map is simply the curvature at that point. This can be seen by observing that the rate at which the tangent vector turns is the same rate at which the normal vector is turning. Moving Δl along the curve, causes the angle of the tangent vector (and the normal vector) to change by $\Delta \theta$. As the length of the piece of curve goes to zero, the ratio

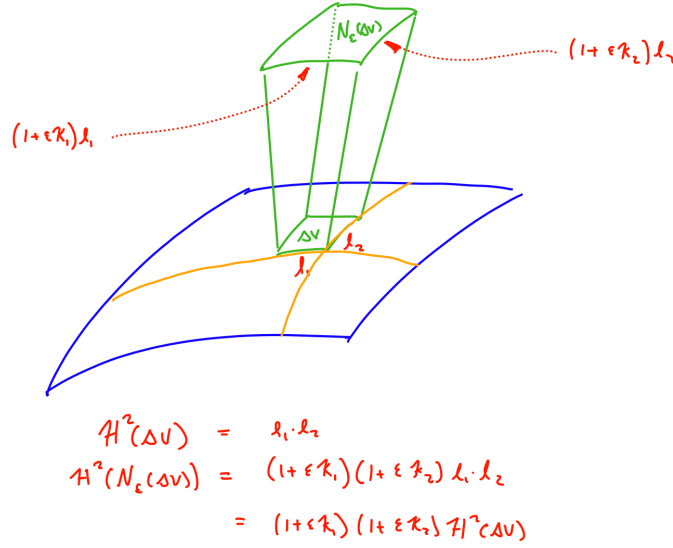


Figure 5: Using the principal curvature directions as a basis, the expansion of the area is given by the product of the dilation in each direction.

$\frac{\Delta\theta}{\Delta l}$ approaches that ratio for motion on the osculating circle. Supposing that the radius of the osculating circle is r , we have that this ratio is $\frac{2\pi}{2\pi r} = \frac{1}{r} = \kappa$, the curvature of the circle.

We conclude for Gauss map $G : M \rightarrow \mathcal{S}^1$, $\det(DG)$ is given by

$$\det(DG) = \kappa$$

See Figure 6.

2.4 Gauss Map for higher dimensional manifolds

In higher dimensions, we may, as we did in the normal flow, find the local coordinates defined by the orthogonal principal curvature directions. In the case, supposing that the surface is $n - 1$ -dimensional in $\mathbb{R}^n$, the determinant of the normal map is simply the product of the determinants of the one dimensional maps in each of the principal directions, i.e. $\prod_{i=1}^{n-1} \kappa_i$ which is the Gauss curvature of the surface.

In Summary: Suppose that M is an $n-1$ dimensional manifold in $\mathbb{R}^n$. Then the determinant of the derivative of the Gauss map $G : M \rightarrow \mathcal{S}^{n-1}$ is given by

$$\det(DG) = \prod_{i=1}^{n-1} \kappa_i.$$

This is also known as the Gauss curvature.

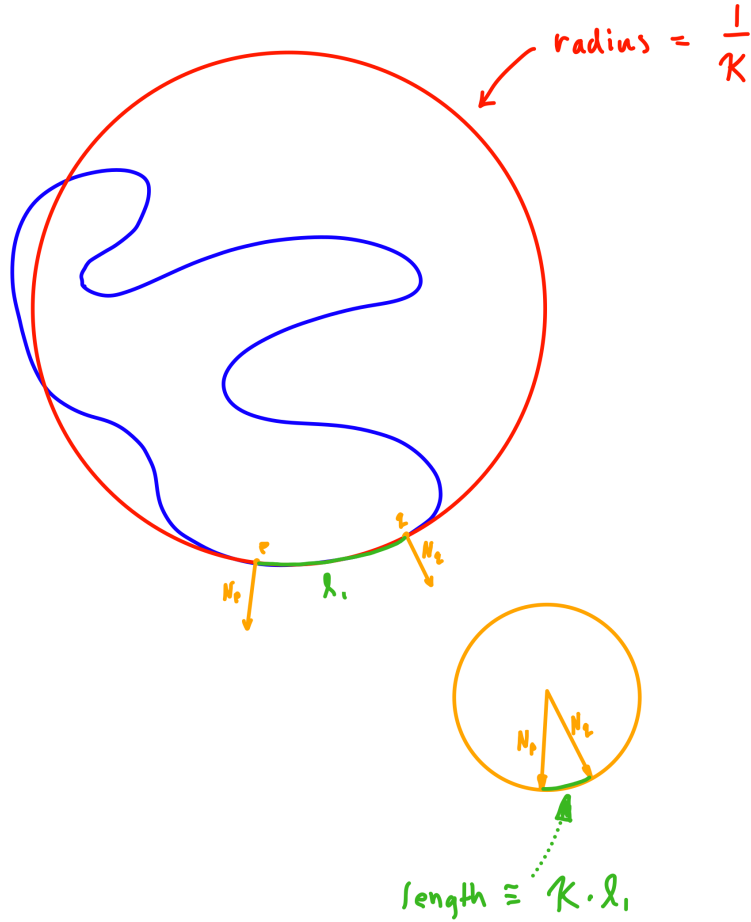


Figure 6: The determinant of the Gauss map for a 1-dimensional curve is simply the curvature of the curve. The normals, shown in yellow are mapped to the unit circle (also shown in yellow).

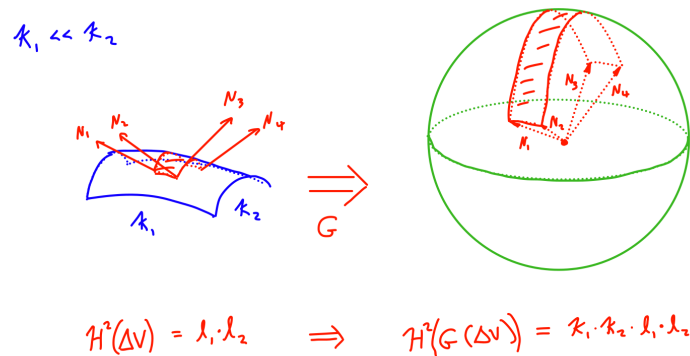


Figure 7: An example in $\mathbb{R}^3$ illustrating the fact that the determinant of the Gauss map is the product of the curvatures. In each of the orthogonal directions, we use the 1-dimensional intuition to get the size of the arc we get on the unit sphere. (Here, the normal vectors are shown in red and the unit 2-dimensional sphere is shown in green.)

Remark 2.2 (Shape Operator). *If you know what the shape operator is ...*

$$S(v) \equiv \frac{dN_x}{dv} \text{ for } v \in T_x M, \text{ where } N_x \text{ is the normal vector at } x$$

... you of course already know the derivative of the Gauss map, because

$$DG = S.$$

Given coordinates on M giving a basis $\{e_i\}_{i=1}^{n-1}$ for $T_x M$, we get that the matrix $[s_{i,j}]$ for S at x is given by $s_{i,j} = \langle S(e_i), e_j \rangle$.

2.5 Up Next ...

To move ahead in our mission to let the normal flow and the Gauss map illuminate all sorts of pieces of nonlinear analysis, we take a closer look at the derivative.

3 The Derivative Path

3.1 Finding the Right Definition

We begin at the beginning, with the usual first definition of derivative that everyone sees first

$$f'(x) \equiv \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

which is valid for functions from $\mathbb{R}$ to $\mathbb{R}$. Let's transform this into something equivalent but much more convenient. Here each line is equivalent to every other line. The last line is the convenient expression we are looking for.

$$\begin{aligned} f'(x) &\equiv \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(x) &= \frac{f(x+h) - f(x)}{h} + k(h) \text{ where } \lim_{h \rightarrow 0} k(h) = 0 \\ (f'(x))h &= f(x+h) - f(x) + h(k(h)) \\ f(x+h) - f(x) &= (f'(x))h + g(h) \text{ where } g(h) \equiv -h(k(h)) \text{ and } \lim_{h \rightarrow 0} \frac{|g(h)|}{|h|} = 0 \end{aligned}$$

3.2 Understanding the Right Definition

Now, recall that every linear function from $\mathbb{R}$ to $\mathbb{R}$ is of the form $L_a : h \rightarrow ah$ for some real number a . Also, any function $g(h)$ that satisfies $\lim_{h \rightarrow 0} \frac{|g(h)|}{|h|} = 0$ is said to be “in little o of h ” and is written $g(h) \sim o(h)$ or $g(h) \in o(h)$

Looking closely at the last line

$$f(x+h) - f(x) = (f'(x))h + g(h)$$

where $g(h) \in o(h)$ we see that this is saying that f is differentiable if and only if

$$f(x+h) - f(x) = L_a(h) + g(h)$$

for some real number a and some function $g(h) \in o(h)$ and that when this is true, the derivative is the map $L_{f'(x)}$. In other words, a map is differentiable at x if and only if it is well approximated by a linear map at x .

Derivatives Done Right: 1-dimensional Case

$$\{f : \mathbb{R} \rightarrow \mathbb{R} \text{ is differentiable at } x\} \Leftrightarrow \{f \text{ is well approximated by a linear function at } x.\}$$

More precisely, f is differentiable if and only if

$$f(x+h) - f(x) = L_a(h) + g(h)$$

where $L_a(h) \equiv ah$ for some real a and $g(h) \in o(h)$. We define $f'(x) \equiv a$. See Figure 8.

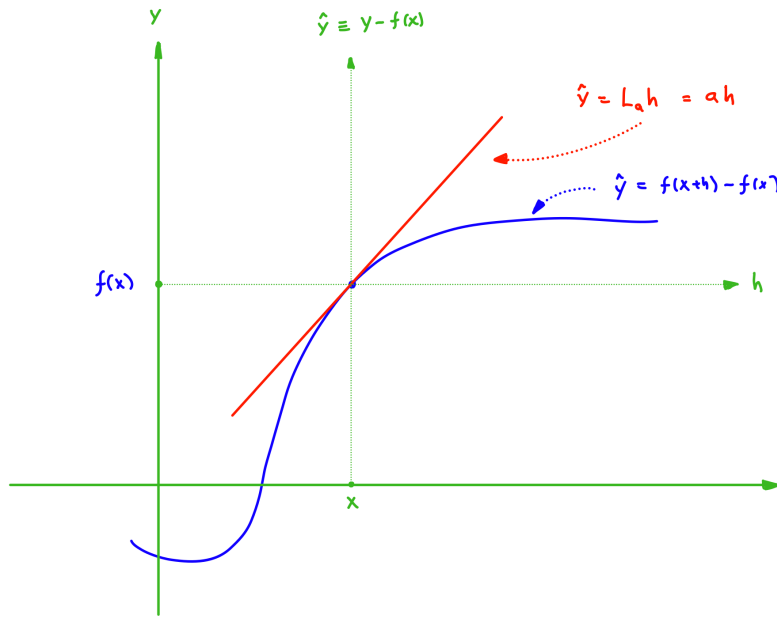


Figure 8: Illustration of the linear approximation of functions from $\mathbb{R}$ to $\mathbb{R}$.

This definition is the right one, because now it is clear that as long as $f : X \rightarrow Y$ where X and Y are vector spaces with norms, we can define the derivative at $x \in X$ to be (if it exists!) the linear approximation to f at x .

Derivatives in Vector Spaces

Suppose $f : X \rightarrow Y$, where X and Y are normed linear (i.e. vector) spaces, with norms $|\cdot|_X$ and $|\cdot|_Y$, respectively. Then we say that f is differentiable at $x \in X$ if there is a linear map $A_x : X \rightarrow Y$ and a function $g : X \rightarrow Y$ with $\lim_{|h| \rightarrow 0} \frac{|g(h)|_Y}{|h|_X} = 0$ (i.e. $g(h) \in o(h)$) such that

$$f(x+h) - f(x) = A_x(h) + g(h).$$

The linear map A_x that makes this true is the derivative. We usually denote A_x by either $D_x f$ or $d_x f$, though sometimes we suppress the x to get Df or df .

3.3 The $D_x f$ vs. $\det(D_x f)$

Remark 3.1. The derivative of $f : \mathbb{R} \rightarrow \mathbb{R}$ is the map $L_{f'(x)} : h \rightarrow (f'(x))h$, not(!) the number $f'(x)$, which is the slope.

So,

Question 3.1. If the slope is not the derivative, what is it?

To see this, we look more closely at what the slope is in the 1-dimensional case. In Figures 9 - 10 we illustrate that the slope gives us both (1) the magnitude of the local change in length under the

mapping and (2) the change in direction or orientation of the pieces of the domain that are getting mapped to the range.

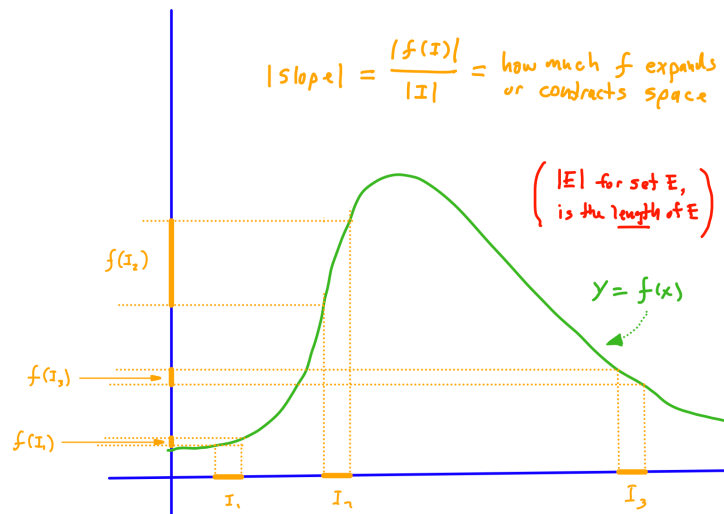


Figure 9: The slope tells us by what fraction the length of a set changes.

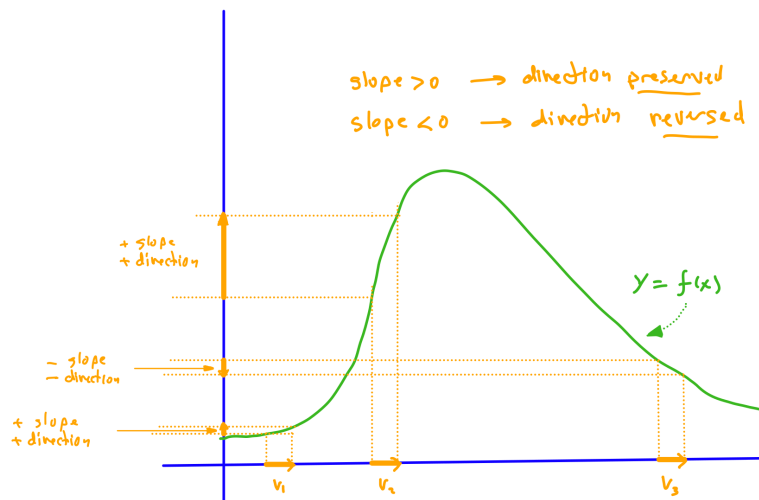


Figure 10: The sign of the slope tells us when the map reverses direction (i.e. orientation).

To move back to the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $n > 1$, recall the singular value decomposition (SVD) for an $n \times n$ matrix (see Figure 11). Using the facts that

- the determinant of a product of matrices is the product of the determinants of those matrices,
- the determinant of an Orthogonal matrix is 1 or -1,
- the determinant of a matrix and its transpose are the same,

Singular Value Decomposition + Determinant

SVD

$$M = \begin{bmatrix} | & | & & | \\ m_1 & m_2 & \dots & m_n \\ | & | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{bmatrix}}_U \circ \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_n \end{bmatrix}}_\Sigma \circ \underbrace{\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}}_{V^T}$$

$\det(M)$

- $M = n \times n$ matrix $\in \mathbb{R}^{n \times n}$
- $M = U \cdot \Sigma \cdot V^T$
- U & V are orthogonal
so $|\det(U)| = |\det(V)| = 1$
- $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$



$$\begin{aligned} \det(M) &= \det(U) \cdot \det(\Sigma) \cdot \det(V^T) \\ &= \pm 1 \cdot \prod_{i=1}^n \sigma_i \cdot \pm 1 \\ &= (\text{signed}) \text{ expansion or contraction factor for sets under } M \\ &= \pm \frac{|M(E)|}{|E|} \quad \text{for } E \subset \mathbb{R}^n \end{aligned}$$

negative determinant $\Rightarrow$ one coordinate has been reflected

Figure 11: The determinant of a matrix tells us about how the linear map scales volume and whether or not it (locally) reverses the orientation of the space as it maps it to the range.

- and a diagonal matrix scales volume by the product of the diagonal entries, i.e. the determinant of the diagonal matrix,

we see that,

The Determinant $\det(D_x f)$ gives Local Scaling and Orientation Changes

When $n > 1$, the number that, like the slope, gives us (1) the scaling of volume and (2) whether or not the orientation is preserved or reversed, is the *determinant of the derivative matrix*, $\det(D_x f)$.

Remark 3.2. *Of course, even when $n = 1$ this is the case because the determinant of an 1×1 matrix is just the single scalar defining that matrix, which in our case, is the slope!*

3.4 Plumbing the Depths of $\int_a^b f' = f(b) - f(a)$

While, when $n > 1$, you may be immediately ready to rewrite the left hand side of the following expression (that makes sense when $n = 1$)

$$\int_a^b f' = f(b) - f(a)$$

as $\int_{\Omega} \det(D_x f) dx$ for some $\Omega \subset \mathbb{R}^n$, the right hand side is probably more challenging. To get a sense for what we might do, we look at the $n = 1$ case in detail. At this point please study Figure 12 very carefully.

Once you understand the figure, it should be pretty clear that what we want for the higher dimensional versions of

$$\int_a^b f' = f(b) - f(a)$$

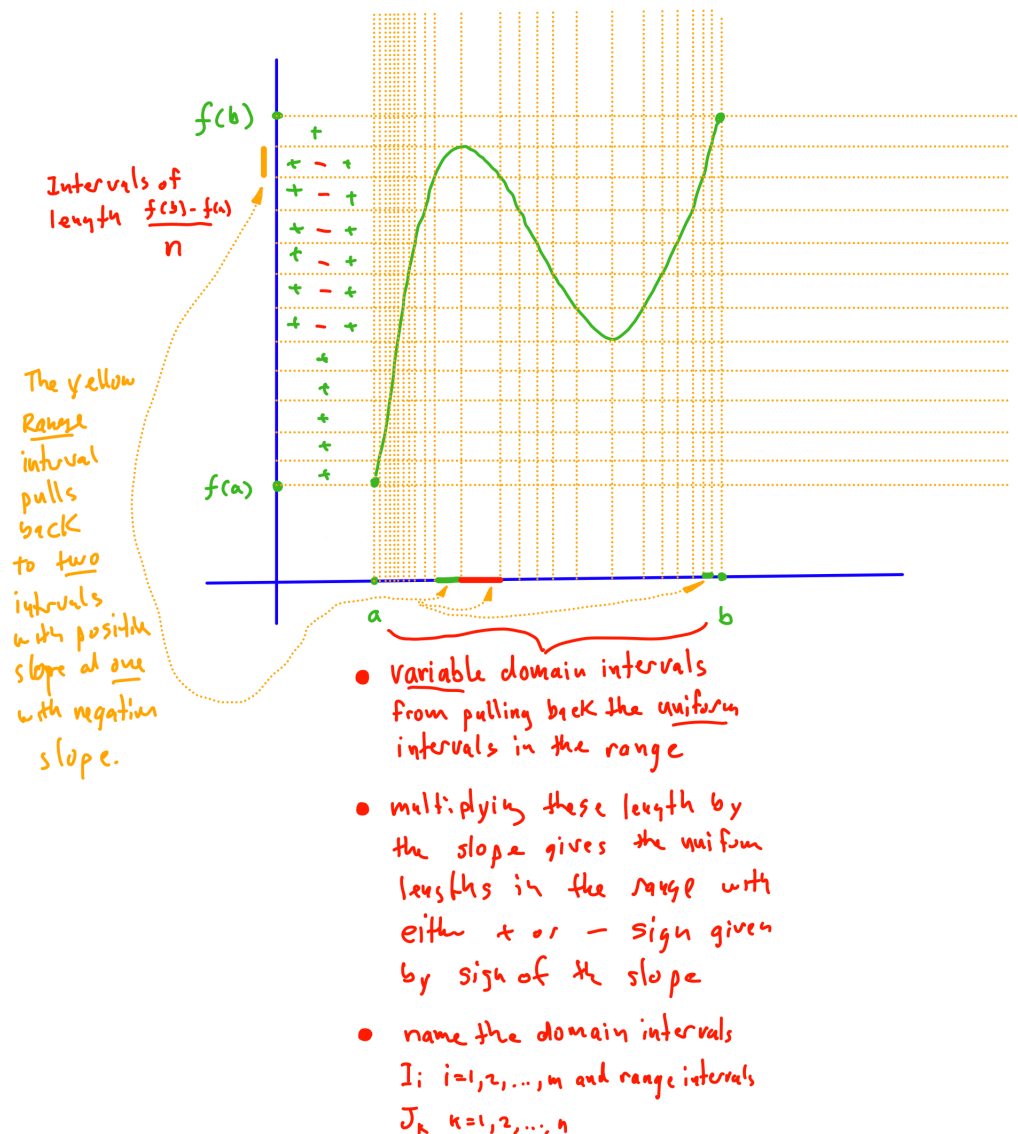
is

$$\int_{\Omega} \det(D_x f) dx = \int_{f(\Omega)} \left(\sum_{x \in f^{-1}(y)} \text{sign}(\det(D_x f)) \right) dy$$

Remark 3.3 (Stokes Theorem). *For those of you who know what forms and exterior derivatives are, you may already know that there is a completely different generalization of $\int_a^b f' = f(b) - f(a)$ called Stokes theorem. In $\mathbb{R}^n$, for any $n - 1$ -form ω and (nice enough) set E with boundary ∂E , we get $\int_{\partial E} \omega = \int_E d\omega$. (It turns out that $\int_a^b f' = f(b) - f(a)$ is just stokes theorem for a 0-dimensional form (i.e. a function) f and set $E = [a, b]$, in $\mathbb{R}$.)*

3.5 Arriving at Two Versions of the Area formula

So, we have arrived at



$$\int_a^b f'(x) dx \cong \sum_{i=1}^m f'(x_i) |I_i| \cong \sum_{k=1}^n \left(\sum_{x_i \in f^{-1}(J_k)} \text{sign}(f'(x_i)) \right) |J_k| \cong \int_{f(a)}^{f(b)} \left(\sum_{x \in f^{-1}(y)} \text{sign}(f'(x)) \right) dy$$

$$\dots \text{ where } \text{sign}(a) = \begin{cases} +1 & \text{if } a > 0 \\ -1 & \text{if } a < 0 \end{cases}$$

(and we ignore $\text{sign}(a)$ when $a=0$)

Figure 12: A different take on integration of the slope. The integral of the determinant of a map f (which in the 1-dim case is the slope) over a set Ω is the same as the integral over $f(\Omega)$ of the {sum of the signs of the derivative determinants on $f^{-1}(y)$ for $y \in f(\Omega)$ }. This will be important in the Gauss Map story!

Alternate Version of the Area formula

$$\int_{\Omega} \det(D_x f) dx = \int_{f(\Omega)} \left(\sum_{x \in f^{-1}(y)} \text{sign}(\det(D_x f)) \right) dy$$

where $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable everywhere (actually, all we really need is that f is Lipschitz, but that is a longer story)

Going back to the figure for the one dimensional case for help, see if you can convince yourself that

The Original Area formula in the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\int_{\Omega} |\det(D_x f)| dx = \int_{f(\Omega)} \mathcal{H}^0(f^{-1}(y)) dy$$

where $\mathcal{H}^0(E)$, the 0-dimensional Hausdorff measure of E , is the number of points in E , $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable everywhere. (Though again, all we need is that f is Lipschitz!)

3.6 Next Stories to Follow

At this point there are two branches to follow forward, as well as the ability to now answer questions in the Normal Flow and Gauss Map sagas. The first branch forward is motivated by the question, Why exactly we can ignore the fact that the sign was not defined when its argument was zero and won't this cause a problem for the "Alternate Version of the Area formula"? This will lead to a degree theory exploration and generalization to n -manifolds. The next branch forward asks, What happens for the "Original Area formula" in the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ when $n > m$, $n < m$, and f is merely Lipschitz? This leads to a more general version of the area formula as well as the co-area formula.

Where To Go from Here At this point there are four tasks or explorations we can move ahead with. Here they are, very succinctly.

Branch Forward Degree Theory

Branch Forward General Area and Co-area Formulas

Gauss Map Saga Integrating the Gaussian Curvature

Normal Flow Saga Evolving the Volume

4 Degree Theory

4.1 A Motivating Exercise

Suppose that f is the function shown in Figure 13. What happens to

$$\sum_{x \in f^{-1}(y)} \text{sign}(f'(x))$$

as you smoothly deform f and avoid y 's where the graph has a zero derivative? Can you deform the function in Figure 13 enough to get the function in Figure 14 without changing that sum? Can you find any smooth deformation at all of f that results in a different value for that sum? (Again, with the assumption that $f'(x) \neq 0$ for all $x \in f^{-1}(y)$)

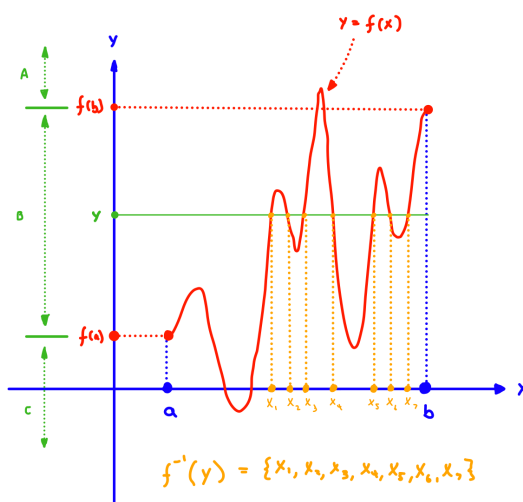


Figure 13: To begin to explore degree theory, we consider the relatively simple, yet informative example of maps from $\mathbb{R}$ to $\mathbb{R}$.

Remark 4.1. *This section will perhaps require a bit more thinking and doodling on paper than the previous sections since there are both more detailed proofs and appeals to geometric intuition that for some readers, is just beginning to form. Anyway, patience and taking time to think and visualize will probably help you push through.*

4.2 Topologies and Equivalence

Now we take a closer look at the somewhat mysterious

$$\sum_{x \in f^{-1}(y)} \text{sign}(\det(D_x f))$$

that we have encountered several times now. It turns out that this is a topological invariant that is not changed by smooth homotopies. ¹

¹While we will (mostly) focus on smooth maps, a deeper look (that we will give references to) extends this quantity to continuous maps (though it is not obvious how to do that from this expression!)

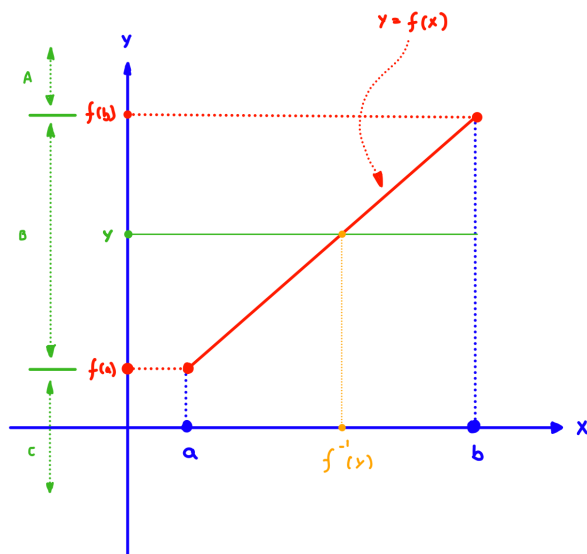


Figure 14: Different function than the function in Figure 13, but with the same values $f(a)$ and $f(b)$

What do I mean by the phrase “not changed by smooth homotopies”?

Beginning at the beginning, recall that one simple definition of continuity is that $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is continuous if you can draw the graph on two dimensional graph paper without lifting the pencil. This definition is intuitively appealing and can easily be turned into the usual $\epsilon - \delta$ definition², but there is a very general definition that is also very simple.

This definition assumes only that you are considering maps between topological spaces. A topological space T , is a space in which you have a class of subsets $\mathcal{T}$ of T called “open” sets. This $\mathcal{T}$ is called the *topology* for T . Any collection of subsets $\mathcal{T}$ of T containing (1) the empty set, (2) the whole space, (3) arbitrary unions of sets in $\mathcal{T}$ and (4) finite intersections of sets in $\mathcal{T}$, is a topology. Now, finally we can get to that very general definition of continuity.

Continuity and Open Sets

A map from one topological space to another, $f : T \rightarrow \tilde{T}$, is continuous if,

$$\tilde{O} \subset \tilde{T} \text{ is open} \Rightarrow f^{-1}(\tilde{O}) \subset T \text{ is open}$$

In other words, the inverse image of open sets are open.

If a mapping f between spaces T and $\tilde{T}$ is (1) invertible (i.e. one to one and onto) and (b) both the map f and its inverse f^{-1} are continuous, then we have a one to one correspondence between open sets. It follows that anything that can be defined in terms of open sets is preserved by f . For example, because connectedness is defined in terms of open sets, both f and f^{-1} map connected

² $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is continuous at x if, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon$. If f is continuous at all x in the domain, we simply say that f is a continuous function.

sets to connected sets. We call such a map a *homeomorphism*. Any property or quantity that is preserved by homeomorphisms is called a **topological invariant**.

There is also a notion of equivalence between maps f and g called **homotopy equivalence**. We will define it for maps that map $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Homotopic Equivalence for Maps

If $h : \Omega \times [0, 1] \rightarrow \mathbb{R}^n$ is continuous and $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$, we say that f and g are *homotopically equivalent*.

In other words, f can be continuously deformed into g .

Notice that we are not assuming invertibility of h (or f and g) and the objects of comparison are maps, not spaces.

We have finally arrived at the notion of invariance that we become involved with as we look for a deeper understanding of the quantity

$$\deg(f, \Omega, y) \equiv \sum_{x \in f^{-1}(y)} \text{sign}(\det(D_x f))$$

where $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Looking Ahead - $\deg(f, \Omega, y)$ is a Homotopy Invariant!

As we will soon see, $\deg(f, \Omega, y)$ is invariant under homotopies – if f and g are homotopically equivalent^a, then

$$\deg(f, \Omega, y) = \deg(g, \Omega, y).$$

^aIt is also necessary to use the extension of the quantity (called the degree of the map at y) to all of $\mathbb{R}^n$ and to disallow homotopies such that $y \in h(\partial\Omega, t)$ for some $t \in [0, 1]$

Remark 4.2 (Metric Spaces). *The zoo of topological spaces is enormous, containing all manner of wild beasts. To tame this a bit, we focus on a narrower collection of topological spaces called metric spaces. A metric space is a topological space, M , with a distance (a metric) ρ , where, for $x, y \in M$, $\rho(x, y)$ is the distance from x and y . This metric is used to generate the topology – in M , a subset O is open if and only if, for any $x \in O$, there is an $\epsilon > 0$ so that $B(x, \epsilon) \subset O$ where $B(x, \epsilon) \equiv \{y \in M \mid \rho(x, y) < \epsilon\}$. These spaces include a rich variety of spaces ranging from $\mathbb{R}^n$ with the usual Euclidean metric (or any other vector space norm) to Hilbert spaces and Banach spaces.*

4.3 Degree Theory in 1 Dimension

Returning now to the motivating exercise above for functions $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, we will now take a much closer look at the calculation of the degree, $\deg(f, [a, b], y)$, using the function shown in Figure 15. It is probably clear to you now that there are two important questions we have not addressed and they are,

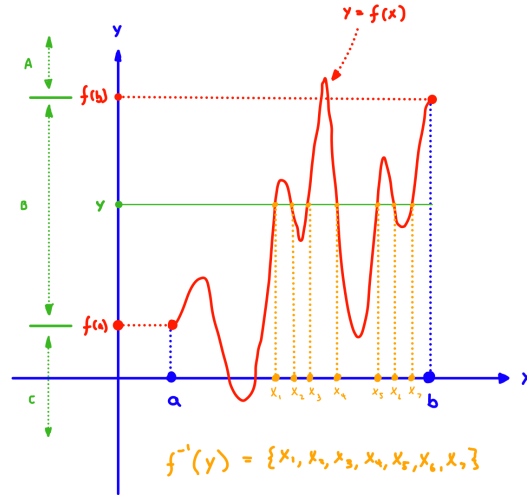


Figure 15: Computing the degree, $\deg(f, [a, b], y)$.

Technical Questions inspired The Definition of Degree of a Map

Question 1 How big is the set of y 's such that $f'(x) = 0$ for some $x \in f^{-1}(y)$

Question 2 Is the sum $\sum_{x \in f^{-1}(y)} \text{sign}(f'(x))$ always a finite sum?

Answering these (and the answer to (1) is very small and (2) is yes!) will free us to ask and answer the homotopy invariance question that was anticipated at the end of the last subsection.

To proceed with answering the Question 1, we need to define what a regular value is.

Regular Values

We define $\mathcal{R}_f$ to be the set of values y such that $x \in f^{-1}(y) \Rightarrow f'(x) \neq 0$. The $y \in \mathcal{R}_f$ are referred to as *regular values* of f .

We now answer to Question 1 by proving a special case of Sard's theorem.

4.3.1 Sard's Theorem for Maps from $\mathbb{R}$ to $\mathbb{R}$

In order to show

Sard's Theorem for $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$

If $f : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$ is differentiable everywhere, then $\mathcal{H}^1(f([a, b]) \setminus \mathcal{R}_f) = 0$

We will use the cone property for derivatives. This is illustrated in Figure 16. The facts in the figure are straightforward consequences of the fact that, if f is differentiable at a , then

$$[f(a+h) - f(a)] - f'(a)h = k(h)h \quad \text{where} \quad \lim_{|h| \rightarrow 0} k(h) = 0.$$

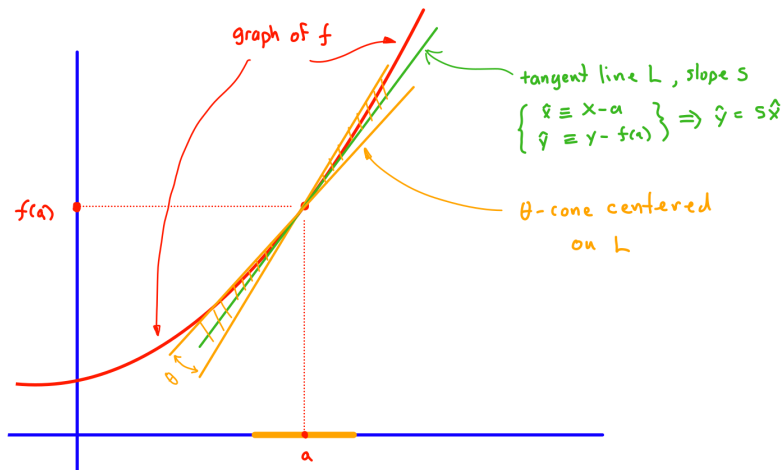


Figure 16: The cone property: If f is differentiable at a and $\theta > 0$, we can choose δ small enough that $I_\delta = [a - \delta, a + \delta]$ is mapped into θ -cone centered on L , the tangent line at a .

Proof of Sard's Theorem in 1 dimension

1. Define $W_f \equiv \{x \in [a, b] | f'(x) = 0\}$.
2. We will prove Sard's theorem by proving that $\mathcal{H}^1(f(W_f)) = 0$.
3. To do that we use the cone property to cover W_f with intervals that shrink as much as we like when mapped to the range.
4. Since the derivative is 0 at each point in W_f , the cone property implies that for any $\alpha > 0$, we can choose a θ such that the aspect ratio of the cone is α and consequently, the symmetric intervals I_δ that map completely inside the cone, satisfy

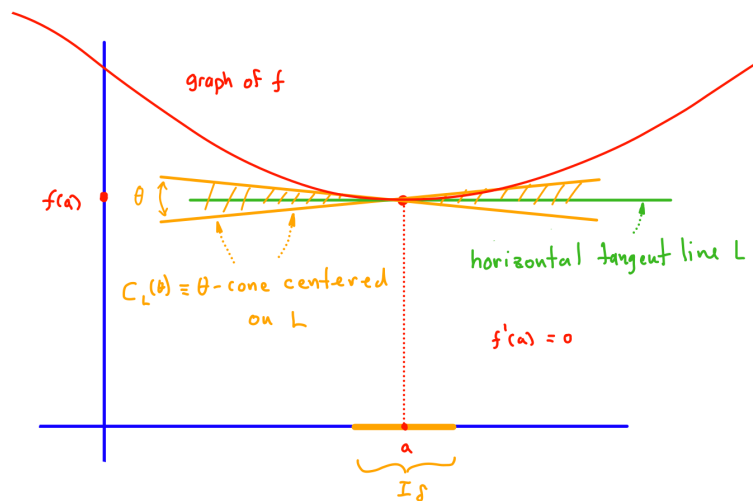
$$\frac{\mathcal{H}^1(f(I_\delta))}{\mathcal{H}^1(I_\delta)} < \alpha.$$

See Figure 17.

5. Use the previous step to find symmetric intervals $I_{\delta_x} = [x - \delta_x, x + \delta_x]$, $\delta_x < 1$, for every point in $x \in W_f$, small enough that the cone with aspect ratio α contains the image of $I_{5\delta_x}$ and consequently, we have

$$\frac{\mathcal{H}^1(f(I_{5\delta_x}))}{\mathcal{H}^1(I_{5\delta_x})} < \alpha.$$

The collection $(C)_f \equiv I_{\delta_x}$'s is a covering of W_f .



- $I_f \equiv \{x \mid |x-a| < \delta\}$
- $f(I_f) \subset C_L(\theta)$
- $\alpha = \frac{H'(f(I_f))}{H'(I_f)} = \tan(\theta) \xrightarrow{\theta \rightarrow 0} 0$

Figure 17: Arbitrarily thin, horizontal cones imply as much shrinkage as you like!

6. Now we use Vitali's covering Lemma for $\mathbb{R}^1$

Vitali's covering Lemma for $\mathbb{R}^1$

Let $\mathcal{C}$ be a family of intervals in $\mathbb{R}$ such that

$$\sup_{I \in \mathcal{C}} (\text{diam}(I)) < \infty.$$

Then there is a countable subcollection $\mathcal{I} \equiv \{I_i\}_{i=1}^N \subset \mathcal{C}$, $N \leq \infty$ such that

(a) $I_i \cap I_j = \emptyset$ when $i \neq j$ and

(b) $\cup_{I \in \mathcal{C}} I \subset \cup_{I \in \mathcal{I}} 5I$

where $5I$ is the interval with the same center as I , but 5 times the diameter. (I recommend Evans and Gariepy's proof of Vitali in the first chapter of their wonderful book [5]. You can also read the proof in the first chapter of Stein's book [19] which is also very highly recommended and should be in the library of everyone working in the proximity of the intersection of analysis and geometry.)

7. So, because we have a uniformly bounded collection of intervals $\mathcal{C}_f$ which cover W_f , we can apply Vitali to get $\mathcal{I}_f = \{I_{\delta_{x_i}}\}_{i=1}^N$, $N \leq \infty$ such that

(a) $I_{\delta_{x_i}} \cap I_{\delta_{x_j}} = \emptyset$ for $i \neq j$ and

(b) $W_f \subset \cup_{I \in \mathcal{C}_f} I \subset \cup_{i=1}^N I_{5\delta_{x_i}}$

8. Calculating, we get that

$$\begin{aligned} \mathcal{H}^1(f(W_f)) &\leq \sum_{i=1}^N \mathcal{H}^1(f(I_{5\delta_{x_i}})) \\ &\leq \alpha \sum_{i=1}^N \mathcal{H}^1(I_{5\delta_{x_i}}) \\ &= \alpha 5 \sum_{i=1}^N \mathcal{H}^1(I_{\delta_{x_i}}) \\ &\leq \alpha 5 \mathcal{H}^1([a, b]) \end{aligned}$$

and because α is arbitrary, we get that $\mathcal{H}^1(f(W_f)) = 0$.

4.3.2 The Sum in question is finite for $y \in \mathcal{R}_f$

Step 1: the Cone Property implies elements of $f^{-1}(y)$ are isolated if $y \in \mathcal{R}_f$

1. Suppose that $x \in f^{-1}(y)$.
2. If $f'(x)$ is not zero, there there is a cone centered on the tangent line and on the point $(x, f(x))$ (i.e. (x, y)), that is skinny enough that the only place the cone intersects the horizontal line defined by the value y , is at the point (x, y) .

3. Because we know that for some perhaps tiny interval centered at x , the entire graph is in the cone, it says that there are no other solutions to $f(x) = y$ in that interval.
4. Thus, when y is a regular value, every $x \in f^{-1}(y)$ is an isolated point of the set $f^{-1}(y)$.

Step 2: Compactness + Step 1 gives us our result

1. Suppose that y is a regular value and that $f^{-1}(y) \subset \Omega$ is an infinite set.
2. Because Ω is compact there must be a point $x^* \in \Omega$ and an infinite sequence $\{x_i\}_{i=1}^\infty \subset f^{-1}(y)$ such that $\lim_{i \rightarrow \infty} x_i = x^*$.
3. But f is continuous, so the fact that $f(x_i) = y$ for all i implies that $f(x^*) = y$.
4. We conclude that $x^* \in f^{-1}(y)$ and x^* is clearly not isolated, which is impossible.

4.3.3 Homotopies that do not move $\partial[a, b]$ leave the degree invariant

Now we return to the question of what happens to the the degree $\deg(f, [a, b], y)$ when we deform f smoothly without changing the values of the function at a and b (i.e. we do not disturb where the boundary of $[a, b]$ gets mapped!).

1. If $f(a) < f(b)$ and $y < f(a)$, we know that every time the curve crosses y downwards it must next cross it moving upwards, so the sum of signs (i.e. $\deg(f, [a, b], y)$) has to be 0 for $y < f(a)$
2. Very similar arguments yield the conclusions that for $f(a) < y < f(b)$, $\deg(f, [a, b], y) = 1$ and for $f(b) < y$, $\deg(f, [a, b], y) = 0$
3. If $f(a) > f(b)$ then for $f(b) < y < f(a)$, $\deg(f, [a, b], y) = -1$
4. If we have $\Omega \subset \mathbb{R}$ that is more complicated - for example $[a, b] \cup [c, d]$ with $a < b < c < d$, we can get more complicated answers for $\deg(f, \Omega, y)$, but the invariance of the degree for homotopies that fix the boundaries of Ω should seem pretty clear at this point.
5. For any $y \in f(W_f) \cap \{\mathbb{R} \setminus f(\partial\Omega)\}$, we define $\deg(f, \Omega, y)$ to be $\deg(f, \Omega, \hat{y})$ for any $\hat{y} \in B(y, \epsilon) \cap \mathcal{R}_f$, where ϵ is chosen small enough that $B(y, \epsilon) \cap f(\partial\Omega) = \emptyset$. This implicitly uses the fact that the degree is constant in the connected components of $\mathbb{R} \setminus f(\partial\Omega)$.

This ends the quasi-rigorous outline of the proof of homotopy invariance of $\deg(f, \Omega, y)$ for differentiable $f : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$ and $y \notin f(\partial\Omega)$.

4.4 Degree Theory in Higher Dimensions

As in the 1-dimensional, we define a *regular value* of a map $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n > 1$ to be a point $y \in f(\Omega)$ such that $\det(D_x f) \neq 0$ for all $x \in f^{-1}(y)$.

4.4.1 Sard's Theorem for Maps from $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n > 1$

In the case of $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ we again get Sard's theorem relatively easily. (We will assume that Ω is compact, implying, among other things, that $\mathcal{H}^n(\Omega) < \infty$.)

Step 1: Balls are compressed into arbitrarily short cylinders

1. The core idea is that, if for a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have $\det(A) = 0$, then the image of $\mathbb{R}^n$ under A , $A(\mathbb{R}^n)$, is contained in some $n - 1$ dimensional subspace $L \subset \mathbb{R}^n$,

$$A(\mathbb{R}^n) \subset L \subset \mathbb{R}^n$$

2. From this it follows that the image of an ϵ ball under A is contained in the intersection of (a) the ball of radius $|A|\epsilon$ centered on the origin³ and (b) L ,

$$A(B(0, \epsilon)) \subset (B(0, |A|\epsilon) \cap L).$$

Note that $(B(0, |A|\epsilon) \cap L)$ is the ball of radius $|A|\epsilon$, centered at the origin, inside L .

3. In the case that f is given by

$$f(a + h) = f(a) + Ah + g(h) \text{ where } g(h) \sim o(h)$$

the above observations yield the facts in Figure 18 where $S = L + f(a)$.

4. Next we observe that since $g(h) \sim o(h)$, we get that

$$\hat{g}(w) \equiv \sup_{|h| < w} |g(h)|$$

is also in $o(w)$. This allows us to use the observational facts recorded in Figure 19 to get that

- (a) $f(B(a, \epsilon)) \subset C_{\delta, \eta} \Rightarrow \mathcal{H}^n(f(B(a, \epsilon))) \leq \omega_{n-1}(|A|\epsilon)^{n-1} \cdot 2\hat{g}(\epsilon)$
- (b) Because $\hat{g}(w) \sim o(w)$, given any $\alpha > 0$ there is an $\epsilon > 0$ such that $\hat{g}(\epsilon) < \alpha\epsilon$.
- (c) So, we get that

$$\begin{aligned} \mathcal{H}^n(f(B(a, \epsilon))) &\leq \alpha \frac{2\omega_{n-1}|A|^{n-1}}{\omega_n} \omega_n \epsilon^n \\ &= \left(\alpha \frac{2\omega_{n-1}|A|^{n-1}}{\omega_n} \right) \mathcal{H}^n(B(a, \epsilon)) \end{aligned}$$

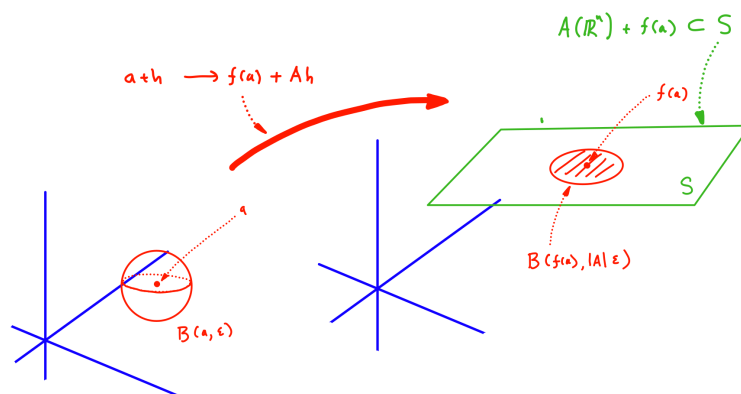
- (d) Since α is arbitrary, we get that $\frac{\mathcal{H}^n(f(B(a, \epsilon)))}{\mathcal{H}^n(B(a, \epsilon))}$ can be made arbitrarily small by choosing ϵ small enough.

Step 2: Apply this to get a cover by balls, each of which compresses by any global factor $0 < \lambda$

1. Choose any $0 < \lambda$.

³ $|A|$ is the operator norm of A and equals the $\sup_{x \in \mathbb{S}^n} |A(x)|$

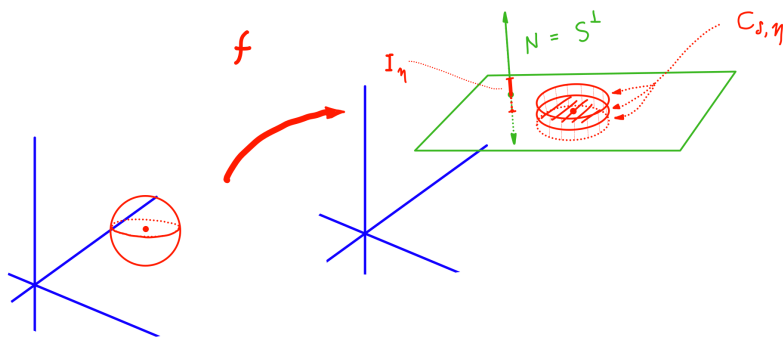
$$f(a+h) = f(a) + Ah + g(h) \quad (\text{and } g(h) \sim o(h)!)$$



- S $n-1$ dimensional affine subspace
- $A(\mathbb{R}^n) + f(a) \subset S$
- $A(B(a, \epsilon)) \subset B(f(a), |A|\epsilon)$ ($|A|$ = operator norm of A)

Figure 18: The linear approximation maps balls into a lower dimensional affine subspace.

$$f(B(a, \epsilon)) \subset C_{\delta, \eta}$$



- $H^1(I_\eta) = 2\eta$, I_η centered on S
- $\hat{g}(w) \equiv \sup_{|h| < w} |g(h)|$ (we know that $\hat{g}(w) \sim o(w)$!)
- $C_{\delta, \eta} = B(f(a), \delta) \times I_\eta$ $\delta = 14|\epsilon + \hat{g}(\epsilon)|$, $\eta = \hat{g}(\epsilon)$
- $f(B(a, \epsilon)) \subset C_{\delta, \eta}$

Figure 19: The fact that the linear approximation differs from the truth by a term that is $o(h)$ permits us to see that the nonlinear map, maps balls into thin cylinders.

2. Now, for each point in $W_f \equiv \{x \in \Omega \mid \det(D_x f) = 0\}$, use Step 1 to choose an $\epsilon_x < 1$ such that

$$\frac{\mathcal{H}^n(f(B(a, 5\epsilon_x)))}{\mathcal{H}^n(B(a, 5\epsilon_x))} < \lambda.$$

3. We now have a collection of balls $\mathcal{C}_f$ which cover W_f with a bound on the diameter of the balls.

Step 3: Use the Vitali covering Lemma to get Sard's Theorem!

Notation: for any ball B , let $5B$ denote the ball with same center as B and 5 times the radius. Now we apply the Vitali covering lemma.

Vitali Covering Lemma in $\mathbb{R}^n$

Let $\mathcal{C}$ be a family of balls in $\mathbb{R}^n$ such that

$$\sup_{B \in \mathcal{C}} (\text{diam}(B)) < \infty.$$

Then there is a countable subcollection $\mathcal{B} \equiv \{B_i\}_{i=1}^N \subset \mathcal{C}$, $N \leq \infty$ such that

1. $B_i \cap B_j = \emptyset$ when $i \neq j$ and
2. $\cup_{B \in \mathcal{C}} B \subset \cup_{B \in \mathcal{B}} 5B$

Apply Vitali to $\mathcal{C}_f$ to get the countable subcollection $\mathcal{B}_f$.

1. We have that $W_f \subset \cup_{i=1}^N 5B_i$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.
2. This implies that

$$\begin{aligned} \mathcal{H}^n(f(W_f)) &\leq \sum_{i=1}^N \mathcal{H}^n(f(5B_i)) \\ &\leq \lambda \sum_{i=1}^N \mathcal{H}^n(5B_i) \\ &= \lambda 5^n \sum_{i=1}^N \mathcal{H}^n(B_i) \\ &\leq \lambda 5^n \mathcal{H}^n(\Omega) \end{aligned}$$

3. Because λ was arbitrary, we conclude that $\mathcal{H}^n(f(W_f)) = 0$.

Regular Values of Differentiable maps

As was the case for $f : \Omega \subset \mathbb{R} \rightarrow \mathbb{R}$, we conclude that if $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable everywhere and $\mathcal{R}_f$ is the set of regular values of f , $\mathcal{H}^n(f(\Omega) \setminus \mathcal{R}_f) = 0$.

Remark 4.3 (This works for Lipschitz maps!). *Actually, the proof we give above for differentiable maps, also works for any Lipschitz map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ because*

1. Rademacher's theorem tells us that F is differentiable everywhere except a set N and $\mathcal{H}^n(N) = 0$.
2. Then, because F is Lipschitz, with some Lipschitz constant $K < \infty$, we know that $\mathcal{H}^n(F(N)) \leq K^n \mathcal{H}^n(N) = 0$.
3. Now we have that F is differentiable for all $x \in D \equiv \mathbb{R}^n \setminus N$ and we can partition D into a countable collection of sets of finite measure, $D = \cup_{i=1}^{\infty} D_i$ with $\mathcal{H}^n(D_i) \leq 1$.
4. For each of these the steps above gives us that $\mathcal{H}^n(F(W_f \cap D_i)) = 0$.
5. Putting all this together, defining $\mathcal{ND}_f$ to be the set of values y , such that at some point in $x \in \{F^{-1}(y)\}$ either
 - (a) F is not differentiable or
 - (b) it is and $\det(D_x F) = 0$
 we get that $\mathcal{H}^n(\mathcal{ND}_f) = 0$.

4.4.2 Finite Sums, Again

The proof that $|\{f^{-1}(y)\}| < \infty$ for $y \in \mathcal{R}_f$ is identical to the proof in the 1 dimensional case.

4.4.3 Homotopy Invariance of Degree in Higher Dimensions

In the 1-dimensional case, it was easy to see that the degree is a homotopy invariance because a line divides the plane into two disconnected pieces. While for the full proof that $\deg(f, \Omega, y)$ is invariant to homotopies that fix $\partial\Omega$ (or even more generally, such that y never encounters the mapped boundary) I will refer you to Deimling's book (see section on references below), we can see that is true for homotopies that are projections of non-self-intersecting deformations.

Follow along with the construction of the transformations by studying Figure 20. In the terminology illustrated in that figure, we can create $H(x, t)$ by composing three maps, the second of which is a function of t ,

$$H(x, t) = P \circ d \circ S = P(d(S(x), t))$$

where $S : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \times \{\mathbb{R} \geq 0\}$, $d : \mathbb{R}^{n+1} \times [0, 1] \rightarrow \mathbb{R}^{n+1}$, and $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ projects out last coordinate, all of these are smooth in all its variables, and all but P is a diffeomorphism onto its image.

An important observation is that for all t , $d(S(\Omega), t)$ is a non-self-intersecting surface and that the set of x that map to y under $H(x, t)$ is given by the intersection of the vertical line through y and the surface generated by $d \circ S$,

$$H^{-1}(\cdot, t)(y) = (d(S(\cdot), t))^{-1} \circ d(S(\Omega), t) \cap \{y \times \mathbb{R}\}.$$

Convince yourself that $D_x H$ alternately reverses and preserves the orientation of $\mathbb{R}^n$ as we move along the vertical line gathering the points in $H(\cdot, t)^{-1}(y)$. (See Section 4.4.4.) So, for the scenario modeled with this composition of functions, it will always turn out that, for all $t \in [0, 1]$, $\deg(H(\cdot, t), \Omega, y) = 0$ for y in Ω^c (complement of Ω) and $\deg(H(\cdot, t), \Omega, y) = 1$ in the interior of Ω .

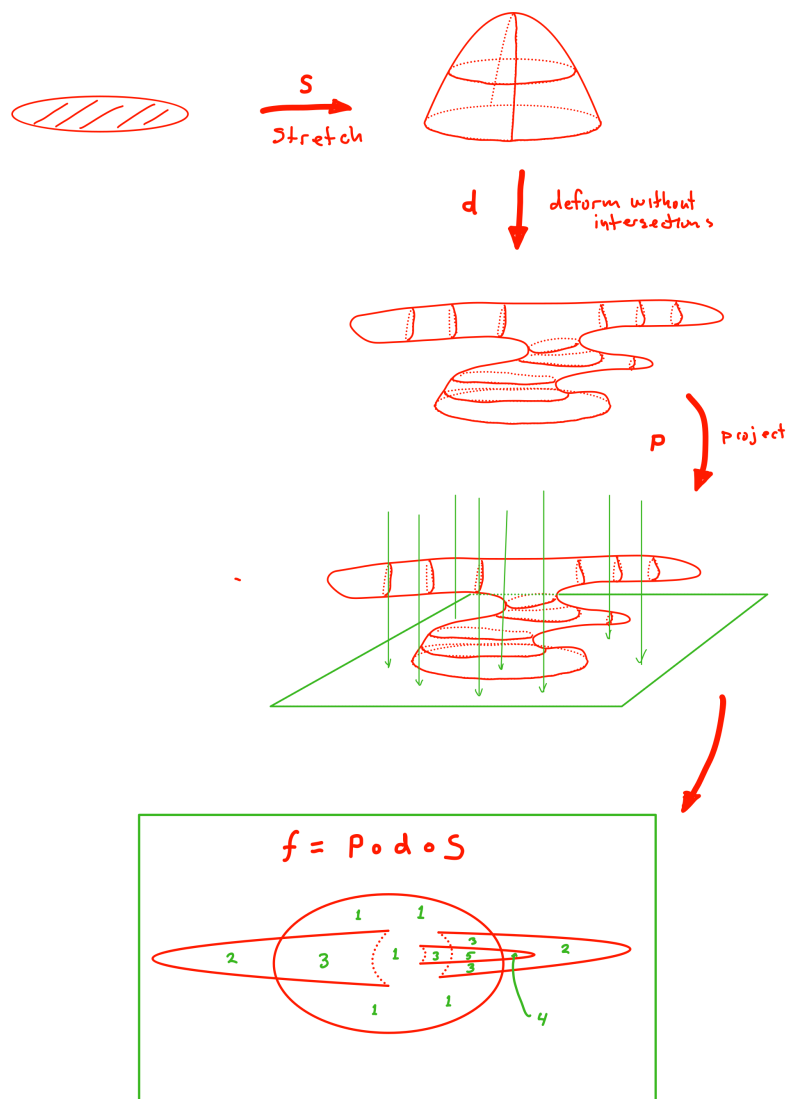


Figure 20: The degree is invariant because the orientation of the domain is alternately preserved and reversed as we trace the points in the preimage of any y . See the explanation in the text above.

In fact, this homotopy invariance is true for *any* homotopy, not just those constructed in the way we illustrated.

Degree Invariance for maps $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$

Suppose that $H : \Omega \times [0, 1] \rightarrow \mathbb{R}^n$ is a smooth homotopy between the maps $g : \Omega \rightarrow \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}^n$, i.e. $H(\cdot, 0) = g$ and $H(\cdot, 1) = f$. Then, for all y in the range space $\mathbb{R}^n$ such that $y \notin H(\partial\Omega, t)$ for any $t \in [0, 1]$, we have that

$$\deg(H(\cdot, t), \Omega, y) \text{ does not depend on } t$$

(Here we have assume that we have extended the degree to values that are not regular as suggested at the end of section 4.3.3 so that the degree is defined for all t on all of the complement of $\cup_{t \in [0, 1]} H(\partial\Omega, t)$)

4.4.4 Orientation and Mappings in Co-Dimension 1

We pause for a brief look at how orientation interacts with mappings of manifolds. In the construction we are giving for the higher dimensional maps between orientable manifolds or pieces of manifolds, we are choosing an orientation (an order to the basis) of points in both the domain and range that can be most simply determined in the following way.

Since we are working with orientable n -dimensional manifolds/surfaces in $\mathbb{R}^{n+1}$, (the manifolds are co-dimension 1) we will always be able to continuously choose a unit normal vector field. At any point x on the manifold M , start with the normal unit vector $N_x \in \mathbb{R}^{n+1}$ and choose a basis of vectors $\{e_i\}_{i=1}^n \subset \mathbb{R}^{n+1}$ for the tangent space of M at x , $T_x M$, such that, expressed in the global basis we have for $\mathbb{R}^{n+1}$, the $n+1 \times n+1$ matrix with columns $[e_1 e_2 \dots e_n N_x]$ has a positive determinant. This process works for any orientable n -manifold in $\mathbb{R}^{n+1}$ ⁴

Now suppose that we have two n -manifolds in $\mathbb{R}^{n+1}$, M_1 and M_2 and a map $F : M_1 \rightarrow M_2$ with tangent map (i.e. derivative) $D_x F : T_x M_1 \rightarrow T_{F(x)} M_2$. Choosing positively ordered⁵ bases for both $T_x M_1$ on the domain manifold and $T_{F(x)} M_2$ on the range manifold, $\det(D_x F) > 0$, tells us that the orientation is preserved, whereas $\det(D_x F) < 0$ indicates the orientation is reversed.

Figure 21 illustrates this for two, 2-dimensional submanifolds of $\mathbb{R}^3$.

To make the calculations simple, we assume that in the vicinity of $F^{-1}(y) = d \circ P^{-1}(y) \subset M$, the

⁴If you are bothered by the fact that at different points of a manifold, these $n+1$ dimensional local bases are not directly comparable, remember that pure rotations (reflection free transformations that preserve lengths) are represented by orthogonal matrices whose determinant is 1. Therefore, any two bases can be transformed onto a standard basis at the origin in $\mathbb{R}^{n+1}$. When we do that either the $(n+1)$ th basis vector will equal $(0, \dots, 0, 1)$ or $(0, \dots, 0, -1)$. Using the fact that the transformation did not change the sign of the determinant of the transformed set of columns, we see that we have a Canonical way to determine of the orientation is consistent over the submanifold.

⁵By “positively ordered”, I mean that the basis $\{e_i\}_1^n$ for M_1 and the basis $\{\hat{e}_i\}_1^n$ for M_2 are both ordered so that $[e_1, e_2, \dots, e_n]$ and $[\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n]$ are positive orientations of M_1 and M_2 , respectively.

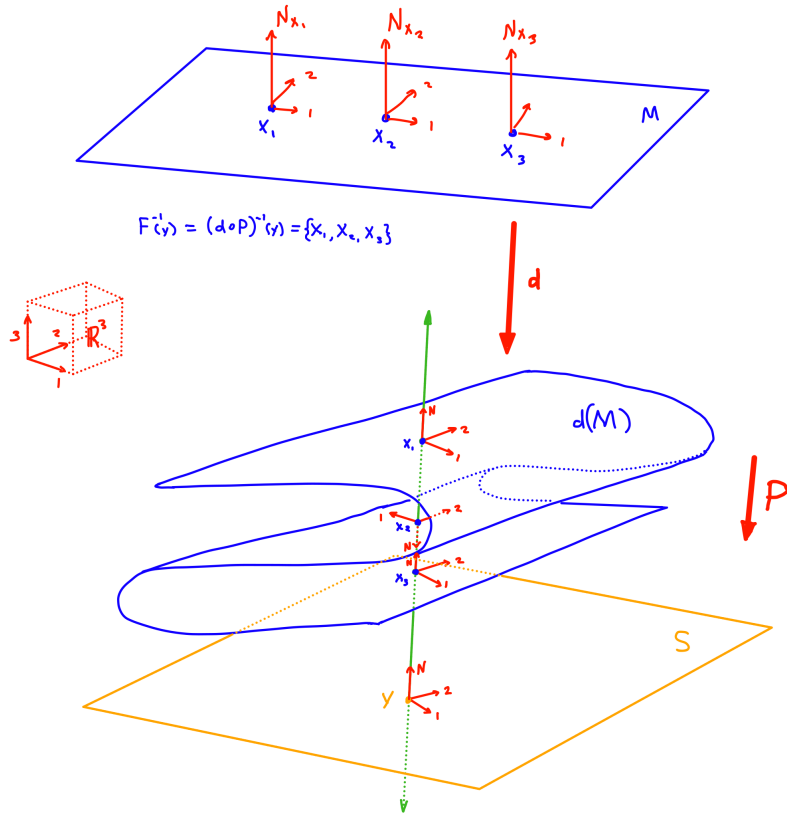


Figure 21: In this figure a 2-manifold $M \subset \mathbb{R}^3$ (in blue) is projected onto a piece of another 2-manifold $S \subset \mathbb{R}^3$ (in yellow).

derivatives are isometries. It is clear from the figure that

$$D_{x_1}F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{x_2}F = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_{x_3}F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (1)$$

So that

$$\text{sign}(D_{x_1}F) = 1, \quad \text{sign}(D_{x_2}F) = -1, \quad \text{and} \quad \text{sign}(D_{x_3}F) = 1$$

and we get that $\sum_{x \in F^{-1}(y)} \text{sign}(\det(D_x F)) = 1$

The fundamental point here is that projections which encounter a point where the normal is pointing away from the target manifold preserve orientation while projections which encounter a point where the normal is pointing towards the target manifold reverse orientation.

Remark 4.4. *When the sub-manifolds we are mapping have no boundary, they divide space into a bounded component and an unbounded component, and we see that at every regular point the projection generates an alternating sequence where the beginning and ending maps both preserve the orientation. So, we have that the degree is 1 for all the regular (and by the extension already discussed, all) points.*

4.4.5 Manifolds: Degree for maps between manifolds.

In these notes, we are especially interested in the Gauss map, so we are interested in homotopy invariance of the degree for maps between manifolds that are diffeomorphic to the Sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. For these manifolds, which we will assume are C^2 , I illustrate the simple intuition that the degree is 1 everywhere. For the full proof of homotopy invariance of degree for maps between manifolds, I refer the reader to Milnor's book [15] (see the comments below, in Section 4.5).

Remark 4.5. *Because we want the Gauss map to be differentiable, we need the normal field (which is the first derivative of the embedding mapping the Sphere to the manifold) to be differentiable in order to use the arguments we have made so far to conclude degree invariance. We therefore require the map mapping the sphere to M to be C^2 .⁶ But because of the observation above that we really only need Lipschitz for our results to go through, we really only need that the embedding map for M is $C^{1,1}$ - i.e. the derivatives exist and are Lipschitz continuous. In fact, as already also noted, degree theory works for maps that are merely continuous, so even if M is merely C^1 , it works!*

We begin by illustrating, in Figure 22, a simple map $f = P \circ d \circ S$, built by smoothly deforming $\mathbb{S}^1$ and then projecting to $\mathbb{S}^1$, (i.e. constructed via the method used in Section 4.4.3). The projection step now projects along the normal field to the sphere. (This is also the closest point projection.) Again, at every regular value y , we have an odd number of elements in $f^{-1}(y)$. Since we want the generated map to be homotopically equivalent to the identity, the diffeomorphism $d \circ S$ must preserve orientation. This implies that the degree of the map is 1 everywhere.

Since it is clear by inspection that we can generate a homotopy $\hat{H} : \mathbb{S}^1 \times [0, 1] \rightarrow \mathbb{R}^2$ such that $\hat{H}(\cdot, 0) = I_{\mathbb{S}^1}$ and $\hat{H}(\cdot, 1) = d \circ S$, adding the projection P gives us the overall homotopy $H \equiv P \circ \hat{H}$ such that $H(\cdot, 0) = I_{\mathbb{S}^1}$ and $H(\cdot, 1) = P \circ d \circ S$.

Observe that the degree of both endpoint maps of the homotopy is 1 everywhere.

⁶Recall that a function or mapping is said to be " C^k " or "in C^k " if it has k continuous derivatives.

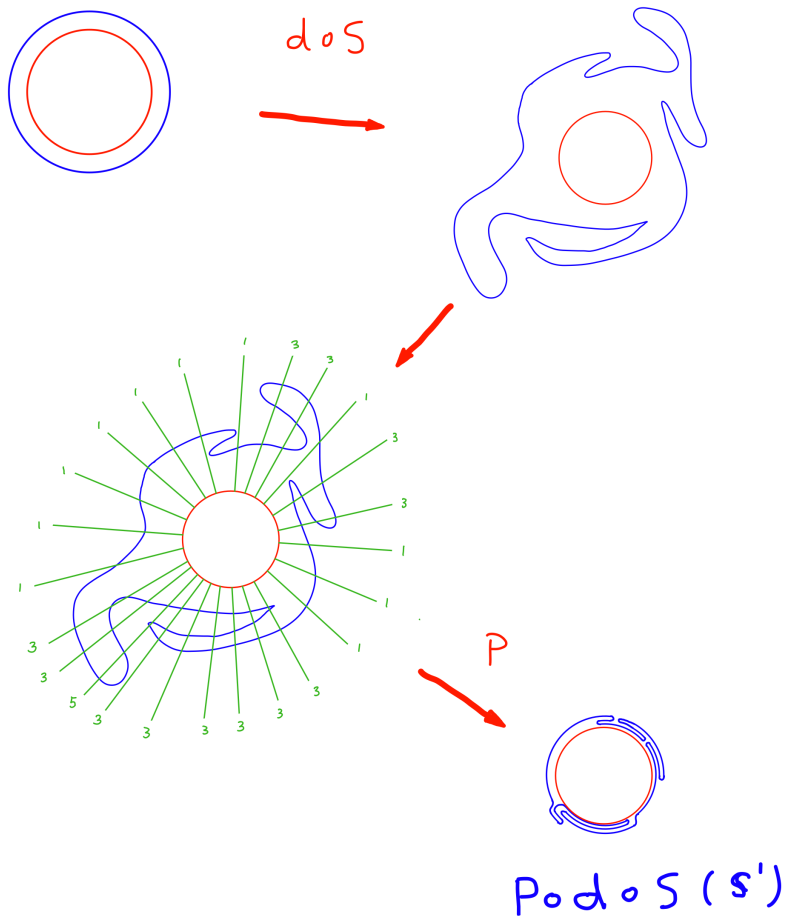


Figure 22: Here we illustrate a map generated as suggested in the last section, but now for maps between Spheres (in this case $\mathbb{S}^1$). It is easy to see that, except at values in the target manifold ($\mathbb{S}^1$) that are not regular values, there is an odd number of crossings of the blue curve to get to the red sphere from outside the blue curve. Thus the degree is 1 almost everywhere and of course this can be exploited to define it to be 1 everywhere.

In the second figure, Figure 23 we begin with the embedded manifold used in the first figure, $M = d \circ S(\mathbb{S}^1)$, and compute the Gauss map for this manifold. This is more intricate, but again, there are an odd number of preimages in $G^{-1}(y)$ for every regular value y and since we have that both the starting $\mathbb{S}^1$ (and therefore M) and the target $\mathbb{S}^1$ are oriented in the usual counterclockwise direction, we get that the degree of this map is 1 at all y excluding the 14 points where the curvature reverses sign (i.e. where the determinant of the Gauss map is 0). After the extension encountered several time already, we have that the degree is 1 for all $y \in \mathbb{S}^1$.

Now, because $d \circ S$ is an orientation preserving diffeomorphism, we have that $G \circ d \circ S$ and G have the same degree for all $y \in \mathbb{S}^1$. And because (1) it is not too hard to see that we can create a homotopy from $I_{\mathbb{S}^1} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ to $G \circ d \circ S : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ and because (2) the degree of the identity map is clearly 1 everywhere, the previous observations imply again that this homotopy preserves the degree.

4.5 References on Degree Theory

The best reference I know of is Deimling's book, now published by Dover, titled "Nonlinear Functional Analysis". While, when I worked through the first chapter, I had the opinion I could improve the exposition, what I have written here does not replace that reference and I recommend his first chapter without reservation for an exposition of the development of degree for maps that are merely continuous.

For degree on manifolds, there is Chapter 5 of Milnor's beautiful "Topology from the Differentiable Viewpoint". This beautiful little book (only 64 pages!) also contains a proof of Sard's theorem for smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, for any integers n and m .

4.6 Next Steps

The next steps for this thread are taken up again in Section 6, where we use the alternate version of the area formula as discussed in Section 3.5 as well as the content of Sections 2.3-2.4 to prove the Gauss Bonnet theorem for $n - 1$ -manifolds in $\mathbb{R}^n$ which are homotopically equivalent to the $n - 1$ -sphere. (It is then easy to see that what we prove there applies to any smooth 2-manifold in $\mathbb{R}^3$.)

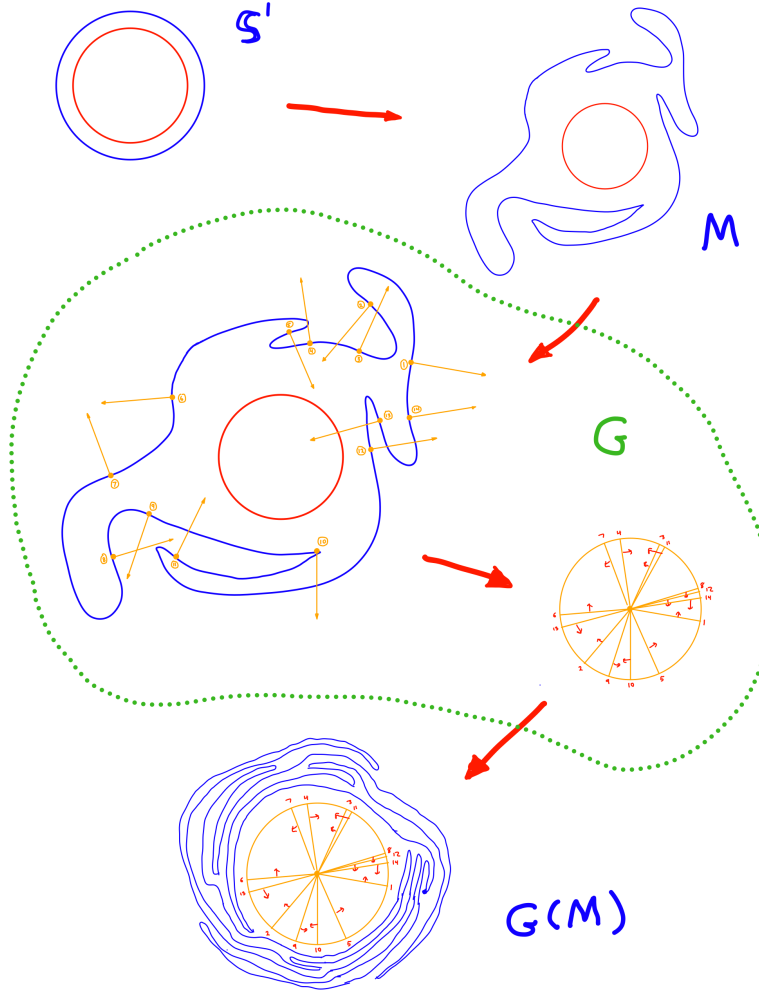


Figure 23: We use the same manifold generated as a part of the last figure, $M = d \circ S(\mathbb{S}^1)$, to illustrate the Gauss map which maps each point of M to its outer unit normal. As can be seen, there are seven intervals of negative curvature and seven intervals of positive curvature, making for a fairly complex looking correspondence, shown in blue in the last part of the figure, between the blue points in M and the yellow target manifold, $\mathbb{S}^1$. The lines in the yellow unit sphere, show where the curvature reverses sign. One can easily verify again that the path from outside the blue to the unit sphere always generates an odd number of crossings (except at points that are not regular values of the map, where we use the extension by approximation encountered over and over already).

5 The Co-area Formula: Lipschitz Maps, Rectifiability, and Calculus in the Wild

We now take a much closer look at the area formula and its generalization to the case in which we are mapping from a higher dimension to a lower dimension. As a part of this, we also take a short tour through the wild lands that open up when we relax the constraints on the mappings and sets we do calculus on. By looking at the titles of the subsection of this section, you already know that those are *Lipschitz Mappings* and *Rectifiable Sets*.

Here are some examples of very natural reasons to study these objects instead of the smooth ones you are probably accustomed to.

Minimal Surfaces Plateau’s problem is the problem of finding a k -dimensional surface spanning a given $k - 1$ -dimensional boundary in $\mathbb{R}^n$. While the first theoretical results (for the case in which the spanning set was assumed to be a mapping of a k -dimensional disk) were given by Douglas in the 1930’s (and for which he received the first Fields medal), the definitive results for much more general surfaces were obtained independently in three separate works in 1960/61. Ennio De Giorgi used the idea of *sets of finite perimeter*, Federer and Fleming used the *integral currents* which Reifenberg used sets with finite k -dimensional Hausdorff measure that were locally, topologically flat $\mathcal{H}^k$ -almost everywhere. The first two depended rather explicitly on rectifiable sets and Lipschitz maps.

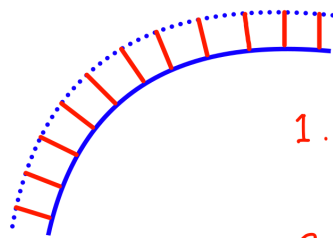
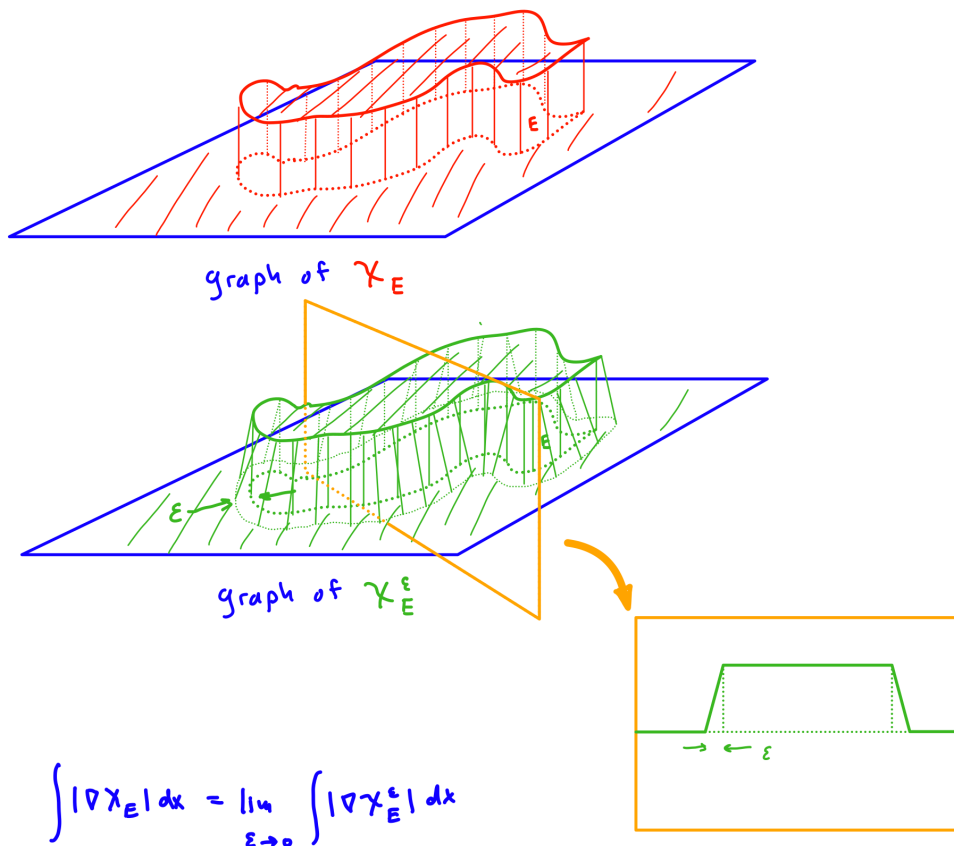
Image Analysis In 1992, Rudin, Osher and Fatemi (ROF) introduced the use of total variation into a variational image denoising functional

$$\min_u F(u) \equiv \int_{\Omega} |\nabla u| dx + \lambda \int_{\Omega} |u - f|^2 dx$$

where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ represented the noisy grayscale image, $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ was the “denoised” image representation and λ was chosen to let the denoised image differ from the noisy data by an amount consistent with the known noise levels. Two authors, Alliney and Nikolova, independently, and in the context of discrete data, considered using an L^1 data fidelity instead of the L^2 term used by ROF. Chan and Esedoglu were the first to study the functional in its continuum form,

$$\min_u F(u) \equiv \int_{\Omega} |\nabla u| dx + \lambda \int_{\Omega} |u - f| dx.$$

Both of these functionals use the total variation term $\int_{\Omega} |\nabla u| dx$ which immediately brings in sets of finite perimeter, which are sets whose measure theoretic boundary (the boundary that integration “sees”) is a rectifiable set of finite length. Therefore, in order to really understand what is going on with minimizers we have to go quite deeply into the theory of rectifiable sets. Figure 24 is an intuitive explanation why the total variation calculates the length of the boundary of a set E when we compute the total variation of χ_E , $\int |\nabla \chi_E| dx$. The key idea (intuitively) is that we can approximate the characteristic function with functions that converge to the characteristic function, but whose gradients we can integrate. We integrate in the direction orthogonal to the boundary and then integrate the result around the boundary to get the length of the boundary.



1. integrate over red normal directions
2. then integrate those results along the ∂E

$$\int |\nabla \chi_E^\epsilon| dx = \int_{\partial E} \left[\int_{\text{normal}} \frac{1}{\epsilon} dx \right] d\mathcal{H}^1 = \int_{\partial E} \left[\frac{1}{\epsilon} \cdot \epsilon \right] d\mathcal{H}^1 = \mathcal{H}^1(\partial E)$$

Figure 24: The total variation of the characteristic function of the set E (having finite perimeter!) returns the length of the (reduced) boundary.

Divergence Theorem One can ask, “How wild can ∂E , the boundary of $E \subset \mathbb{R}^n$, be and still be able to make sense of

$$\int_{\partial E} \vec{v}(y) \cdot \vec{n}(y) d\mathcal{H}^{n-1}y = \int_E \nabla \cdot \vec{v}(x) dx$$

where $\vec{v}$ is a vectorfield?” The answer, again is, it can be the measure theoretic boundary of a set of finite perimeter, which again, is a rectifiable set.

Remark 5.1. *The $n - 1$ dimensional boundary that is typically used in the case of a n -dimensional set of finite perimeter is the reduced boundary, but this is always only different from the measure theoretic boundary on a set with $n - 1$ -dimensional measure 0. And because there is a decent chance somebody can guess a definition of measure theoretic boundary, I use that in this quick introduction to the problems. See Chapter 5 of Evans and Gariepy’s [5] for all the details.*

All of these lead very naturally to the focus on Lipschitz maps and rectifiable sets. In the process of studying those, we often need to understand how the measure of sets is mapped around, so we find it very natural to use the area and co-area formulas in that study.

But we will also find that rectifiable sets by themselves present fascinating challenges, some of which are, in fact, answered by the Lipschitz map/rectifiable set version of the co-area formula. We begin our tour aimed at this wilder side of calculus by recalling the area formula for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

5.1 Review of the Area Formula for $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

Recall, from Section 3.5, that the area formula for maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ is given by

The Original Area formula in the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\int_{\Omega} |\det(D_x f)| dx = \int_{f(\Omega)} \mathcal{H}^0(f^{-1}(y)) dy$$

where $\mathcal{H}^0(E)$, the 0-dimensional Hausdorff measure of E , is the number of points in E , $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable everywhere. (Though again, all we need is that f is Lipschitz!)

We also know that in the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, if we remove the absolute value from the determinant, we instead get

Alternate Version of the Area formula

$$\int_{\Omega} \det(D_x f) dx = \int_{f(\Omega)} \left(\sum_{x \in f^{-1}(y)} \text{sign}(\det(D_x f)) \right) dy$$

where $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable everywhere (again, all we really need is that f is Lipschitz).

which, we now write more simply as

Mapping Degree Version of the Area formula

$$\int_{\Omega} \det(D_x f) dx = \int_{f(\Omega)} \deg(f, \Omega, y) dy$$

where $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz.

It turns out though that the area formula works just as well when $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n \leq m$.

5.2 Extension of the Area Formula to $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n < m$

In the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we need something like $|\det(D_x f)|$ that measures the expansion and contraction of volume under mapping, but of course, the determinant will not work because now $D_x f$ is not a square matrix. But, this is no problem because once again, the SVD illuminates the way. The idea is again that because

$$D_x f = U \circ \Sigma \circ V^t$$

(where V is an $n \times n$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix and U is an $m \times m$ orthogonal matrix), if $D_x f$ is full rank we know that $D_x f$ is mapping the volume from n dimensions to an n -dimensional subspace of $\mathbb{R}^m$ and the expansion or contraction is given by the product of the σ 's down the diagonal of Σ . We see that

$$\begin{aligned} D_x f^t \circ D_x f &= V \circ \Sigma^t \circ U^t \circ U \circ \Sigma \circ V^t \\ &= V \circ \Sigma^t \circ \Sigma \circ V^t \\ &= V \circ \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) \circ V^t \end{aligned}$$

which implies that this product of the diagonal entries of Σ is just

$$J_x f \equiv \sqrt{\det(D_x f^t \circ D_x f)}$$

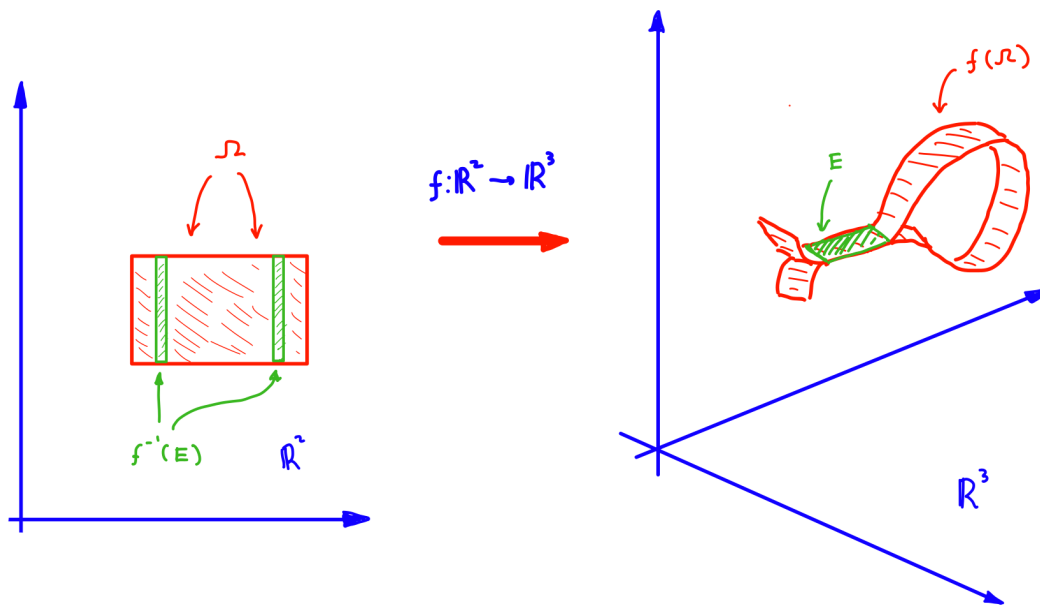
so that now we have

Area formula for $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n \leq m$

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth, $\mathcal{H}^0(E)$, the 0-dimensional Hausdorff measure of E , is the number of points in E , $\Omega \subset \mathbb{R}^n$, and $J_x f \equiv \sqrt{\det(D_x f^t \circ D_x f)}$, then

$$\int_{\Omega} J_x f dx = \int_{f(\Omega)} \mathcal{H}^0(f^{-1}(y) \cap \Omega) d\mathcal{H}^n y$$

Figure 25 illustrates this in the case that $n = 2$ and $m = 3$.



- $\mathcal{H}^0(f^{-1}(y)) = 2$ if $y \in E$
- $\mathcal{H}^0(f^{-1}(y)) = 1$ if $y \in f(\Omega) \setminus E$

Figure 25:

So far, this should not be very new - given what you learned in the sections on derivatives and degree, you could guess all these things. But what about the case in which $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$? It turns out that this is a very interesting, very powerful case that opens up subtle things having to do with rectifiable sets which we will also introduce.

5.3 The Co-area Formula: the case of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$

When we are mapping from a higher to a lower dimensional space, the derivative will have a non-trivial null space, so the level sets $f^{-1}(y)$ for $y \in f(\Omega)$ will be $n - m$ -dimensional submanifolds of $\mathbb{R}^n$ whenever y is a regular value of a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. (i.e. whenever f is at least C^1 and $D_x f$ is full rank for every $x \in f^{-1}(y)$.) That this is true should seem reasonable since N_x , the null space of $D_x f$, is (1) $n - m$ dimensional, (2) tangent to the level sets and (3) varies smoothly as we move around the domain. Figure 26 illustrates this for a map from $\mathbb{R}^3$ to $\mathbb{R}$. We gather then, that, for $D_x f$ full rank, $D_x f(N_x \times N_x^\perp) = D_x f(N_x^\perp) = \mathbb{R}^m$ and that the expansion or contraction that $D_x f$ imposes on $\subset N_x^\perp$ can be extracted from

$$\begin{aligned} D_x f \circ D_x f^t &= U \circ \Sigma \circ V^t \circ V \circ \Sigma^t \circ U^t \\ &= U \circ \Sigma \circ \Sigma^t \circ U^t \\ &= U \circ \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2) \circ U^t \end{aligned}$$

where now Σ is an $m \times n$ matrix, and U and V are, respectively, $m \times m$ and $n \times n$ orthogonal matrices. So, the quantity we want giving the expansion or contraction – the product of the diagonal

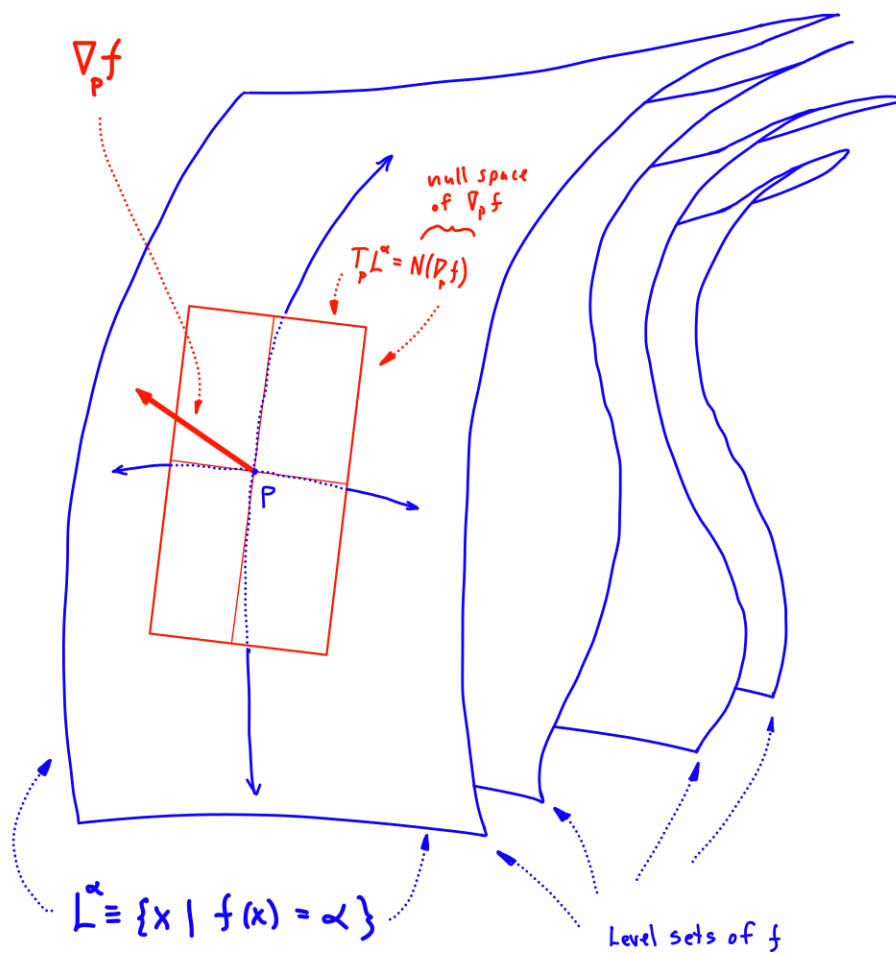


Figure 26: The smooth mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ has level sets $L^\alpha = f^{-1}(\alpha)$ that are submanifolds whenever the derivative is full rank at every point in L^α . In this case, full rank is equivalent to $D_p f = \nabla_p f \neq 0$.

entries of Σ – is just

$$J_x f \equiv \sqrt{\det(D_x f \circ D_x f^t)}$$

so that we are now ready to state several versions of the co-area formula.

Co-area formula for smooth $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$: Version I

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth, $\Omega \subset \mathbb{R}^n$, and $J_x f \equiv \sqrt{\det(D_x f \circ D_x f^t)}$, then

$$\int_{\Omega} J_x f \, dx = \int_{f(\Omega)} \mathcal{H}^{n-m}(f^{-1}(y) \cap \Omega) \, dy$$

This is illustrated in Figure 27. The details in the figure are important since they lead you to the key insights behind the proof of the theorem. Briefly (do study the figure!) ...

1. The inverse image of a small m -cube in the range – in the figure the range is $m = 1$, and 1-cubes are intervals I_y – is, $f^{-1}(I_y)$, a band around the level set of y , where we are defining $I_y = [y - \epsilon, y + \epsilon]$.
2. In the directions orthogonal to the level set, the derivative expands or contracts the volume so that all these orthogonal slices map forward to I_y . That is, for every $x \in f^{-1}(y)$, $J_x f \mathcal{H}^1(L_x) = \mathcal{H}^1(I_y)$.
3. And so, if we integrate $J_x f$ over the little band $f^{-1}(I_y)$, doing this first in the direction orthogonal to the level set and then that result along the level set (a sort of gently nonlinear Fubini's theorem style of integration) we get that the result is $\mathcal{H}^1(f^{-1}(y) \cap \Omega) \cdot \mathcal{H}^1(I_y)$. (In the general case we would have $\mathcal{H}^{n-m}(f^{-1}(y) \cap \Omega) \cdot \mathcal{H}^m(C_y)$ for a tiny m -cube C_y .)

The full version of the smooth co-area formula, which allows us to integrate g over the whole domain or just over level sets (separately) and put the results together in the range, is given by

Co-area formula for $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$: Version II

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth, $g : \Omega \rightarrow \mathbb{R}$ is integrable, $\Omega \subset \mathbb{R}^n$, and $J_x f \equiv \sqrt{\det(D_x f \circ D_x f^t)}$, then

$$\int_{\Omega} g(x) J_x f \, dx = \int_{f(\Omega)} \left(\int_{f^{-1}(y) \cap \Omega} g(x) \, d\mathcal{H}^{n-m} x \right) dy$$

It turns out that all this works perfectly well for the cases in which f is Lipschitz and we are mapping from an n -rectifiable set to an m -rectifiable set. So we now look at Lipschitz maps and Rectifiable sets after which we state the co-area formula (it is essentially the same) for the Lipschitz case and discuss several interesting implications.

5.4 Lipschitz Maps

A Lipschitz map is not only continuous, but also bounded in terms of how far things are stretched. To be more quantitative, here is the precise definition.

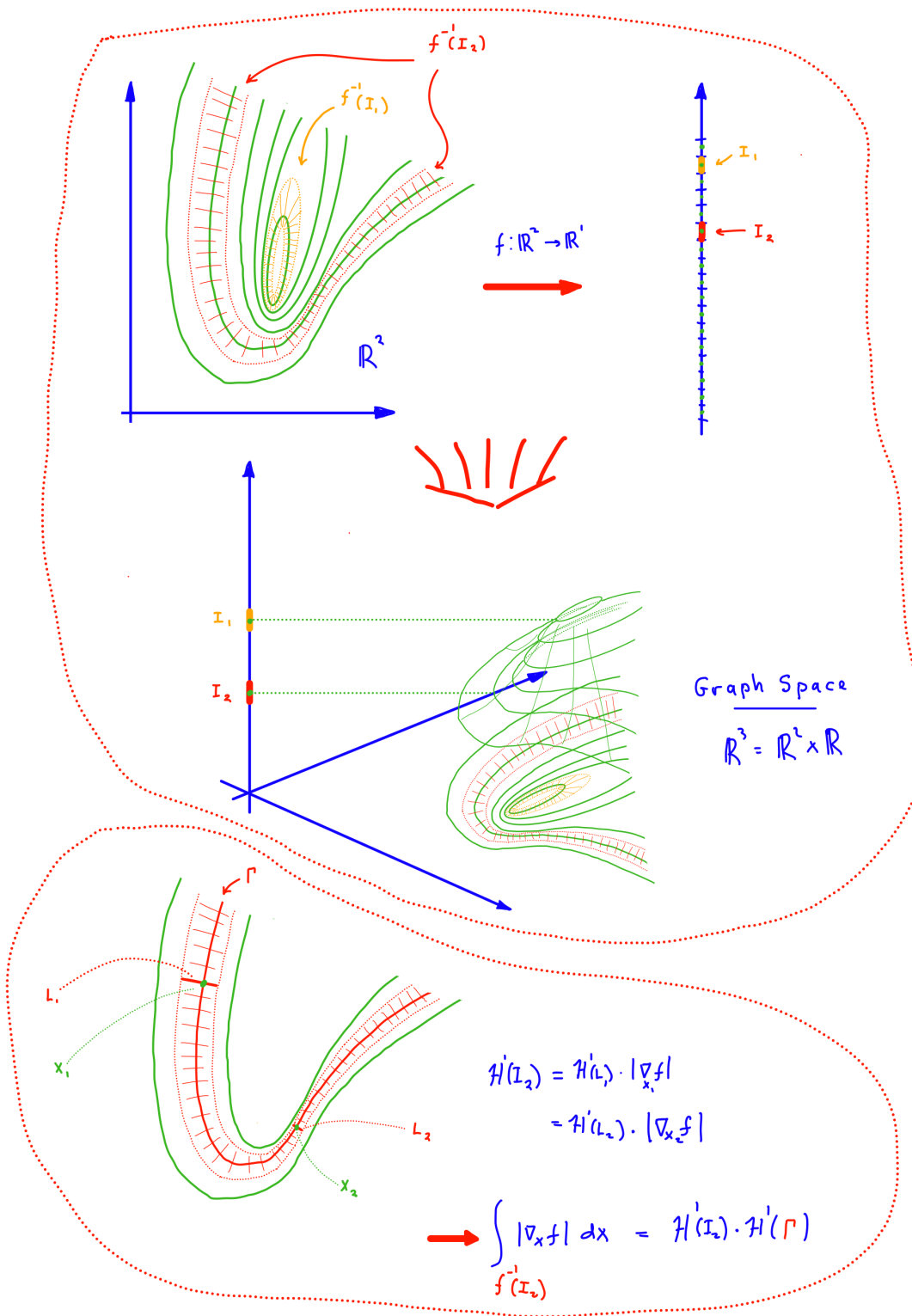


Figure 27: A visual, intuitive idea of the proof of the smooth co-area formula.

Lipschitz Continuous Maps

Suppose that f maps a metric space X into another metric space Y . Then we say that f is Lipschitz continuous if there is a constant $K < \infty$ such that

$$\rho_Y(f(x), f(y)) \leq K \rho_X(x, y) \quad \forall x, y \in X$$

where ρ_X and ρ_Y are the metrics on X and Y respectively.

In these notes $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ so f is Lipschitz if and only if $|f(x) - f(y)| \leq K|x - y| \quad \forall x, y \in \Omega$, where the norms on each side are the Euclidean norm of the range (left side) and domain (right side).

Lipschitz functions can be non-smooth as can be seen in Figure 28 because the only thing that is necessary is that all secant line slope magnitudes $\frac{|f(y) - f(x)|}{|x - y|}$ be bounded by some $K < \infty$. In fact,

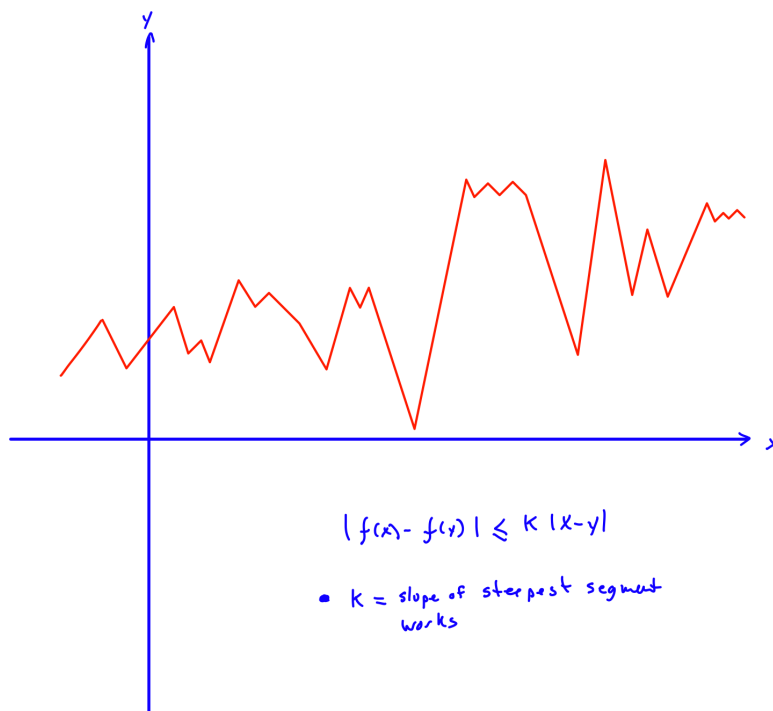
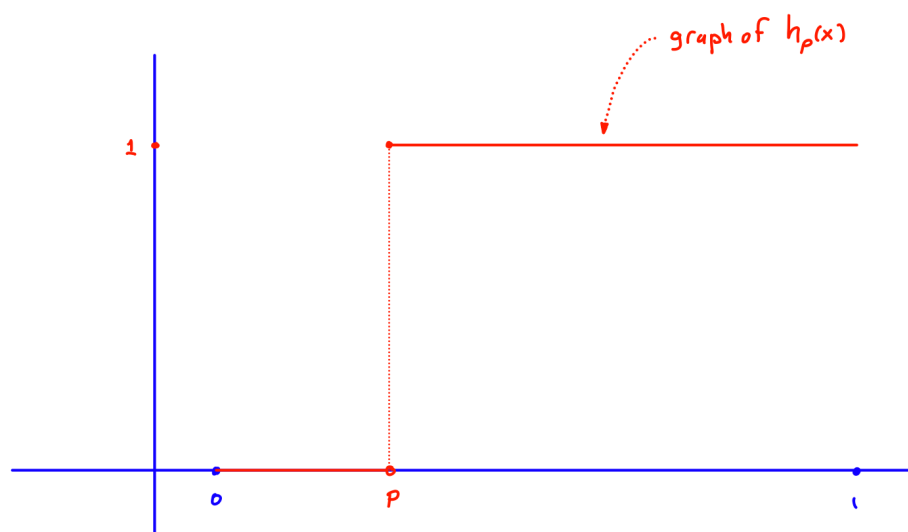


Figure 28: Lipschitz does not mean smooth!

with a little work, we can create an example of a Lipschitz $f : [0, 1] \rightarrow \mathbb{R}$ that is not differentiable on a dense subset of $[0, 1]$! See Figure 29 for one such construction.

But Rademacher's Theorem tells us that no matter how hard we try, we cannot introduce so many kinks and turns that the points in the domain where f fails to be differentiable has positive measure!

Because of this key result, we immediately know that the left hand side of the co-area formula can be calculated since $J_x f$ is defined almost everywhere! What about the right hand side of the formula? A key result tells us almost every level set of $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$ is an $n - m$ -rectifiable



- $\{q_i\}_{i=1}^{\infty} = \mathbb{Q} \cap (0,1)$... the rational points in $(0,1)$

- $g(x) \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} h_{q_i}(x)$... $g(0) = 0$ $g(1) = 1$

- $f(x) \equiv \int_0^x g(y) dy$

Facts: • $|f(x) - f(y)| \leq |x - y| \quad \forall x, y \in [0,1]$

- f is not differentiable at any q_i which are dense in $[0,1]$

- f is differentiable $\forall x \in [0,1] \setminus \{q_i\}_{i=1}^{\infty}$

Figure 29: This is really an implicit exercise - prove the statements made in the figure!

set. We will show this to be true later, as a result of something we are calling “Almost Sard’s Theorem” that follows from the co-area formula (see Section 5.7.2), though it should already seem plausible that the level sets minus points in those sets where the derivative does not exist or is not full rank should be manifold like.

But before we get there, we have to explain what a rectifiable set is, give the more general area and co-area formulas and discuss the ramifications of the theorem including the things the theorem asserts about Lipschitz maps.

5.5 Rectifiable Sets

We finally come to the definition of rectifiability

Rectifiable Sets: Version I

A set $M \subset \mathbb{R}^n$ is said to be k -rectifiable if

$$M \subset \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \cup N_0^k$$

where each of the f_i ’s are Lipschitz maps from $\mathbb{R}^k$ into $\mathbb{R}^n$ and $\mathcal{H}^k(N_0^k) = 0$.

It is a theorem that the following definition is equivalent

Rectifiable Sets: Version II

A set $M \subset \mathbb{R}^n$ is said to be k -rectifiable if

$$M \subset \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^k) \cup N_0^k$$

where each of the f_i ’s are C^1 maps from $\mathbb{R}^k$ into $\mathbb{R}^n$ and $\mathcal{H}^k(N_0^k) = 0$.

See Figure 30 for examples of rectifiable sets.

Remark 5.2. *We often work under the assumption that a k -rectifiable set M also satisfies $\mathcal{H}^k(M) < \infty$ or even that M is both closed and $\mathcal{H}^k(M) < \infty$.*

As mentioned in previous sections, because of the “almost-everywhere differentiable” property, many results – including the area and co-area formulas – also hold for Lipschitz maps. There is huge amount in and around the theory of rectifiable sets in $\mathbb{R}^n$ (and on manifolds and in metric spaces ...). The references in Section 8 include places to go to begin the exploration of these ideas.

Before moving to the generalized co-area formula and its very interesting implications, we mention one very important theorem.

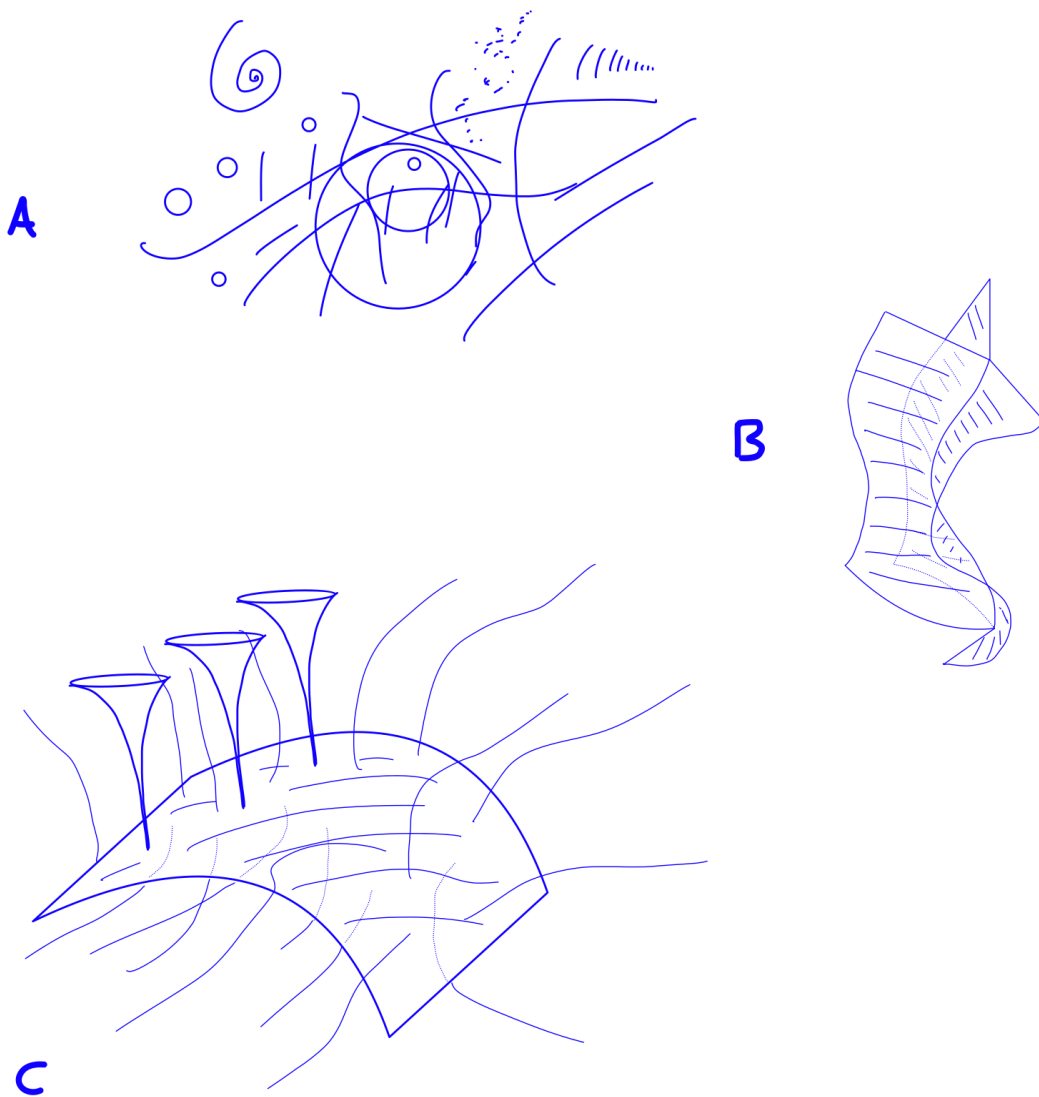


Figure 30: Here are three examples of rectifiable sets. A is 1-rectifiable set in $\mathbb{R}^2$ and both B and C are 2-rectifiable sets in $\mathbb{R}^3$. In B , every point in the depicted set, except those that are in the 1 dimensional “hairs” (which are in the N_0^2 for that set) have approximate tangent planes.

Rectifiability = measure-theoretically flat

A set $M \subset \mathbb{R}^n$ satisfying $\mathcal{H}^k(M \cap K) < \infty$ for any compact $K \subset \mathbb{R}^n$ is k -rectifiable if and only if it has approximate k -dimensional tangent planes at $\mathcal{H}^k$ -almost every point in M

What is an approximate tangent plane?

Approximate Tangent Planes

We will say that $M \subset \mathbb{R}^n$ has the k -dimensional subspace $T \subset \mathbb{R}^n$ as an approximate tangent plane at the point p if

$$\int_{\eta_{p,r}(M)} \phi(x) d\mathcal{H}^k x \rightarrow_{r \rightarrow 0} \int_T \phi(x) d\mathcal{H}^k x$$

where

$$\eta_{p,r}(M) = \frac{M - p}{r}.$$

and ϕ is an compactly supported C^∞ function mapping $\mathbb{R}^n$ to $\mathbb{R}$.

This takes a bit to grok⁷ if you have not already seen it. The thing to focus your effort on is understanding the map $\eta_{p,r}$. This is the zooming into M at the point p that you already, intuitively, know has to be central to the definition of a tangent approximation.

Approximate Tangent Planes, Intuitively

The basic idea is that if we shift the point p to the origin and then dilate the set by dividing by r and let r go to 0, we will zoom into M at p . If the changing value of the integral on the left converges to the integral on the right, and does this no matter what “test function” ϕ we choose, it says that, in the limit, as $r \rightarrow 0$, whatever difference there is between the zoomed in set and the plane T is invisible to integration over those sets ... the set differences are invisible “measure theoretically”.

To master the idea of approximate tangent planes, convince yourself that, in the case of the rectifiable set **C** shown in Figure 30, the 2-dimensional approximate tangent planes exist at all points except (1) at the boundary points (the three circles from the cusps and the boundary of the “bent” rectangle) and (2) at points on the 1-dimensional lines that are away from the cusps and away from the bent rectangle.

In the last two subsections of this section on area and co-area formulas, we will explore rectifiability a bit more and make a few notes on the references that go much deeper. Now, on to the general area and co-area formulas.

⁷From the Wikipedia article on grok: Grok is a neologism coined by the American writer Robert A. Heinlein in his 1961 science fiction novel *Stranger in a Strange Land*. The Oxford English Dictionary summarizes the meaning of grok as “to understand intuitively or by empathy, to establish rapport with”, and “to empathize or communicate sympathetically (with); also, to experience enjoyment”

5.6 The Area and Co-area Formulas in the nonsmooth case

Now that we have a bit of a feel for what Lipschitz maps and rectifiable sets are like, we can write down the more general versions which are essentially the same as the other ones you have already seen. In the next section we will explore the non-trivial implications of having the co-area formula hold true for maps that are merely Lipschitz.

Area formula for $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n \leq m$ - Nonsmooth Case

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $\mathcal{H}^0(E)$, the 0-dimensional Hausdorff measure of E , is the number of points in E , $\Omega \subset \mathbb{R}^n$, $g : \Omega \rightarrow \mathbb{R}$ is $\mathcal{H}^k$ -integrable and $J_x f \equiv \sqrt{\det(D_x f^t \circ D_x f)}$, then

$$\int_{\Omega} g(x) J_x f \, dx = \int_{f(\Omega)} \left(\sum_{x \in f^{-1}(y) \cap \Omega} g(x) \right) d\mathcal{H}^n y$$

noting that in the case that $g = 1$, we recover

$$\int_{\Omega} J_x f \, dx = \int_{f(\Omega)} \mathcal{H}^0(f^{-1}(y) \cap \Omega) d\mathcal{H}^n y$$

Co-area formula for $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, $n > m$ - Nonsmooth Case

If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, $g : \Omega \rightarrow \mathbb{R}$ is integrable, $\Omega \subset \mathbb{R}^n$ and $J_x f \equiv \sqrt{\det(D_x f \circ D_x f^t)}$, then

$$\int_{\Omega} g(x) J_x f \, dx = \int_{f(\Omega)} \left(\int_{f^{-1}(y) \cap \Omega} g(x) d\mathcal{H}^{n-m} x \right) dy$$

noting that in the case that $g = 1$, we recover

$$\int_{\Omega} J_x f \, dx = \int_{f(\Omega)} \mathcal{H}^{n-m}(f^{-1}(y) \cap \Omega) dy$$

The very general case where we are mapping between rectifiable sets, and we use min and max functions to combine the two formulas into one, is given by

Area/Co-area formula for $f : \Omega \subset M_k \rightarrow M_l$

If $f : \Omega \subset M_k \rightarrow M_l$ is Lipschitz, M_k and M_l are, respectively, k -rectifiable and l -rectifiable subsets of $\mathbb{R}^n$, $g : \Omega \rightarrow \mathbb{R}$ is $\mathcal{H}^k$ -integrable and $J_x f \equiv \sqrt{\det(D_x f \circ D_x f^t)}$, then

$$\int_{\Omega} g(x) J_x f \, d\mathcal{H}^k x = \int_{f(\Omega)} \left(\int_{f^{-1}(y) \cap \Omega} g(x) d\mathcal{H}^{\max(0, k-l)} x \right) d\mathcal{H}^{\min(l, k)} y$$

5.7 Examples and Notes on Calculus in the Wild

When we open the door to functions which are merely Lipschitz continuous and sets or spaces that are merely rectifiable, we are opening the door to an enormous amount of beautiful complexity and wonder. Those that have seen this panorama unfold in front of them, are never quite the same (in a good way!). But the work to see this is such that this vision probably only emerges as a result of a very serious immersion into this new realm - it is not something that can be transmitted easily.

As far as I am able to see, the only thing writing like this (that you are reading) can do is to invite someone to dive in, to commit to a deep exploration, to a decision to find that vision for themselves.

In this last section before the references, I will start with two simple examples and then point out a beautiful Sard's Theorem like conclusion for Lipschitz mappings from the co-area formula. After that I look at an eclectic sampling from this new "calculus in the wild" universe we have opened up.

5.7.1 Two Examples

While applications of the area and co-area formulas abound, perhaps the first two (that are always encountered in the typical training that all scientists who use calculus) are (1) the areas of graphs and (2) integration of radially symmetric functions.

Areas of Graphs: The Area Formula

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ define $F(x) \equiv (x, f(x)) \in \mathbb{R}^n \times \mathbb{R}$. Now, for any set $\Omega \subset \mathbb{R}^n$, we have that $F(\Omega)$ is the graph of $f(x)$ over Ω . A natural question that is asked and answered in the calculus course that all physical science students take, is, given $\Omega \subset \mathbb{R}^n$ what is the "area" (i.e. the n -volume) of the graph $F(\Omega) \subset \mathbb{R}^{n+1}$? It is ...

$$\mathcal{H}^n(F(\Omega)) = \int_{\Omega} \sqrt{1 + \nabla f \cdot \nabla f} dx$$

Where is the area formula here? Let's write out the area formula for this map

$$\int_{\Omega} J_x F dx = \int_{F(\Omega)} \mathcal{H}^0(F^{-1}(y)) d\mathcal{H}^n y$$

but then we notice that $\mathcal{H}^0(F^{-1}(y)) = 1$ on all of $F(\Omega)$ and so we can simplify to

$$\int_{\Omega} J_x F dx = \int_{F(\Omega)} d\mathcal{H}^n y$$

which simplifies further to

$$\int_{\Omega} J_x F dx = \mathcal{H}^n(F(\Omega))$$

which is the same as

$$\mathcal{H}^n(F(\Omega)) = \int_{\Omega} J_x F dx$$

and we are done if we can show that $J_x F = \sqrt{1 + \nabla f \cdot \nabla f}$. But $D_x F$ is $n + 1$ by n matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ f_{x_1} & f_{x_2} & \cdots & f_{x_n} \end{bmatrix}.$$

Now, calculating $D_x F^t \circ D_x F$ we get

$$D_x F^t \circ D_x F = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} f_{x_1} f_{x_1} & f_{x_1} f_{x_2} & \cdots & f_{x_1} f_{x_n} \\ f_{x_2} f_{x_1} & f_{x_2} f_{x_2} & \cdots & f_{x_2} f_{x_n} \\ \vdots & \vdots & & \vdots \\ f_{x_n} f_{x_1} & f_{x_n} f_{x_2} & \cdots & f_{x_n} f_{x_n} \end{bmatrix}.$$

This can be simplified to

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{bmatrix} \begin{bmatrix} f_{x_1} & f_{x_2} & \cdots & f_{x_n} \end{bmatrix} = I_n + \nabla f \cdot (\nabla f)^t.$$

where the last term is the *outer product* of the gradient and its transpose.

Now we choose an orthonormal basis $\{u_1, u_2, \dots, u_n\}$ such that $u_n = \frac{\nabla f}{|\nabla f|}$, the unit vector in the direction of ∇f . Let U be the orthogonal matrix whose columns are these basis vectors.

$$\begin{aligned} I_n + \nabla f \cdot (\nabla f)^t &= U \circ U^t \circ (I_n + \nabla f \cdot (\nabla f)^t) \circ U \circ U^t \\ &= U \circ (U^t \circ (I_n + \nabla f \cdot (\nabla f)^t) \circ U) \circ U^t \\ &= U \circ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 + |\nabla f|^2 \end{bmatrix} \circ U^t \end{aligned}$$

We therefore conclude that

$$\begin{aligned} \sqrt{\det(D_x F^t \circ D_x F)} &= \sqrt{\det \left(U \circ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 + |\nabla f|^2 \end{bmatrix} \circ U^t \right)} \\ &= \sqrt{(\det(U))(1 + \nabla f \cdot \nabla f)(\det(U^t))} \\ &= \sqrt{1 + \nabla f \cdot \nabla f} \end{aligned}$$

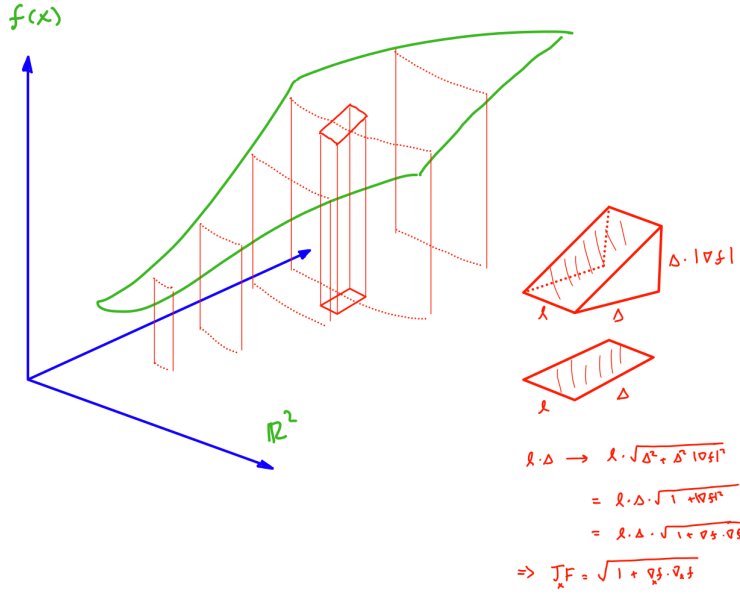


Figure 31: An illustration of the direct geometric calculation verifying $J_x F = \sqrt{1 + \nabla f \cdot \nabla f}$ in the case that $n = 2$

There is an easy geometric route to this fact. Figure 31 is an illustration of this when $n = 2$.

Remark 5.3. We can also get the n -volume of the span of the n columns of $D_x F$ using the generalized Pythagorean theorem (see Chapter 3 of Evans and Gariepy [5]) to arrive at $J_x F = \sqrt{1 + \nabla f \cdot \nabla f}$.

Integrating Radially Symmetric Functions: The Co-area Formula

When integrating a function $f(x)$ over $\mathbb{R}^n$ and f depends only on $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, the integral can be rewritten⁸ as

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\{r \geq 0\}} f(r) n\omega(n) r^{n-1} dr,$$

something quite familiar to anyone who has taken a few courses in physics, though few recognize this as the co-area formula in action.

To see this, we start by noting that we are integrating over the level sets of the distance from the origin $F(x) \equiv d(x, E)$ where $E = \{0\}$. Now we recall or figure out that any distance function has a gradient of length 1 almost everywhere outside of E . So in our case, we get that that $J_x F = 1$ except at the origin. We note also that $F^{-1}(r) =$ the sphere of radius r centered at the origin and

⁸the $\omega(n)$ in this integral is the n -volume of the unit ball in $\mathbb{R}^n$

that $F(\mathbb{R}^n) = [0, \infty)$ so that the co-area formula becomes

$$\begin{aligned}\int_{\mathbb{R}^n} f(x) J_x F dx &= \int_{F(\mathbb{R}^n)} \left(\int_{F^{-1}(r)} f(x) d\mathcal{H}^{n-1} x \right) dr \\ \int_{\mathbb{R}^n} f(x) dx &= \int_0^\infty \left(\int_{\partial B(0,r)} f(x) d\mathcal{H}^{n-1} x \right) dr \\ \int_{\mathbb{R}^n} f(x) dx &= \int_0^\infty f(r) n\omega(n) r^{n-1} dr\end{aligned}$$

where, again, $\omega(n)$ = the n -volume of the unit ball in $\mathbb{R}^n$, (which implies that the $n-1$ -volume of the sphere of radius r in $\mathbb{R}^n$ is $n\omega(n)r^{n-1}$).

Remark 5.4. In general, if $g : \mathbb{R}^n \rightarrow \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $F(x) \equiv d(x, \Omega)$ is the distance from x to Ω , and $E \subset \Omega^c$ we get that

$$\int_E g(x) dx = \int_0^\infty \left(\int_{F^{-1}(s)} g(x) d\mathcal{H}^{n-1} x \right) ds$$

a nonlinear version of Fubini's theorem! In the case that $g(x) = 1$, we get

$$\mathcal{H}^n(E) = \int_0^\infty \mathcal{H}^{n-1}(F^{-1}(s)) ds$$

5.7.2 Almost Sard's Theorem

Now we observe that if we define $A = \{x \mid J_x F = 0\} \cup \{x \mid D_x F \text{ does not exist}\}$ then we get that

$$\begin{aligned}0 &= \int_A J_x F dx \\ &= \int_{F(A)} \mathcal{H}^{n-m}(F^{-1}(y) \cap A) dy \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(F^{-1}(y) \cap A) dy\end{aligned}$$

implying that for any Lipschitz map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and for all but a measure zero set of y in the range space,

$$\mathcal{H}^{n-m}(F^{-1}(y) \cap A) = 0$$

which should be compared to Sard's conclusion that for any smooth map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and for all but a measure zero set of y in the range space

$$F^{-1}(y) \cap A = \emptyset.$$

Remark 5.5. Finally, we can see that because

$$\mathcal{H}^m(\{y \mid \mathcal{H}^{n-m}(F^{-1}(y) \cap A) \neq 0\}) = 0,$$

we know that if w is in the complement of this measure 0 set of “bad values” in the range, the corresponding level set $L_w = F^{-1}(w)$ has at most some junk set N_w (where the either the derivative does not exist or is not full rank) with $\mathcal{H}^{n-m}(N_w) = 0$. Then, for any $x \in L_w$ that is not in N_w , F has a full rank derivative $D_x F$. The null space of the linear operator $D_x F$ is therefore the tangent plane for L_x at x . By the “rectifiability = measure-theoretically flat” fact from above, we conclude that for all the good values w , the level set L_w is an $n - m$ -rectifiable subsets of the domain.

5.7.3 Level sets of the Distance Function

Here is a somewhat surprising (at first glance) fact - the distance function to any compact subset of $\mathbb{R}^n$ has rectifiable level sets for any level $\alpha > 0$!

Figure 32 gives the correct insights you need to prove this. As far as I can remember, either a conversation with Bill Allard, or a note in one of Chazal et al.’s papers, was where I first saw this simple argument. The crux of the matter is that this curved cone, shown in yellow changes continuously as we move around the α level set of the distance function from S_k . We can use this to get that that level set lives in a collection of pieces which are Lipschitz graphs. (There is some work to see this, but what is shown in the figure is enough, with some added work, to get you there.)

5.7.4 A Rectifiable Measure

Figure 33 shows the construction of a “nice” rectifiable measure from a rectifiable set that does not have the local tangent plane property because it is not locally finite. But with the addition of weights (or densities) this set is tamed and now for any point $x \in L_i$, we get an approximate tangent plane with density $\frac{1}{2^i}$. That is

$$\int_{\mathbb{R}^2} \phi(x) d\eta_{p,r}(\mu) \xrightarrow{r \rightarrow 0} \int_{L_i} \phi(x) d\left(\frac{1}{2^i} \llcorner \mathcal{H}^1\right) x$$

where, for any measurable E ,

$$\eta_{p,r}(\mu)(E) \equiv \frac{\mu(rE + p))}{r}.$$

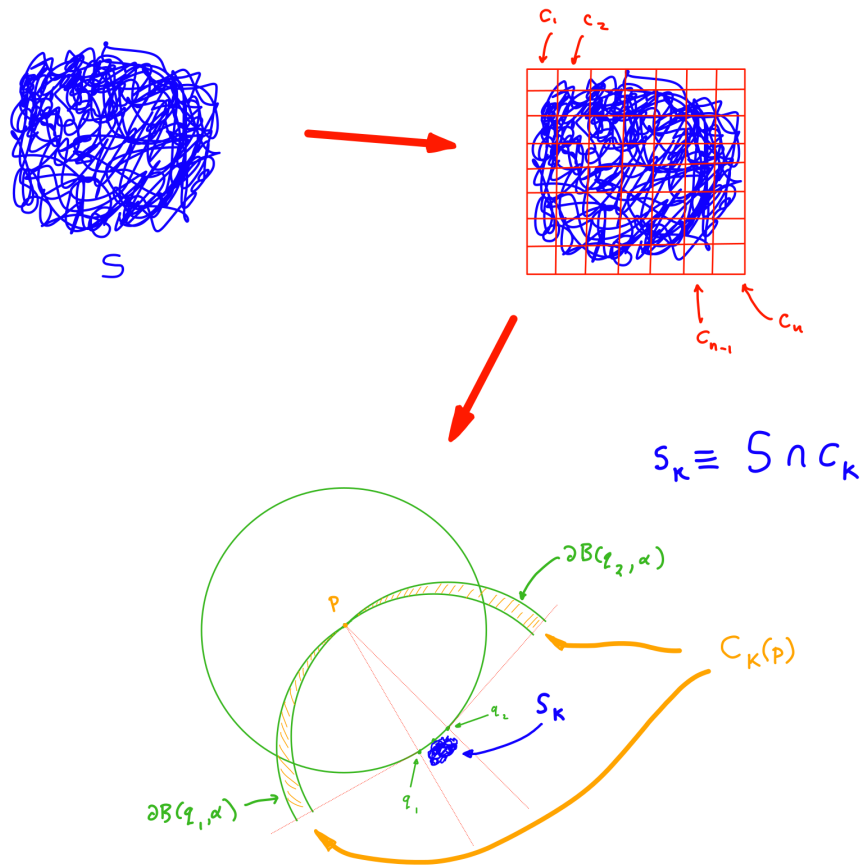
and as before ϕ is an compactly supported C^∞ function mapping $\mathbb{R}^2$ to $\mathbb{R}$.

Remark 5.6 (Tangent Measures and approximate tangent planes). *Suppose that $\{r_i\}_{i=1}^\infty$ converges monotonically to 0 and that α is a non-negative real number. Then a limit of*

$$\eta_{p,r_i}(\mu)(E) \equiv \frac{\mu(r_i E + p))}{r_i^\alpha}$$

is a tangent measure of μ at p . (There can be many different tangent measures at a point in the support of a measure.) In the case that every monotonic sequence generates the same limit measure and that limit is equal to $\beta \mathcal{H}^k \llcorner L$ where L is some k -dimensional subspace of $\mathbb{R}^n$ and β is some positive real number, we say that μ has L as an approximate tangent plane with density β .

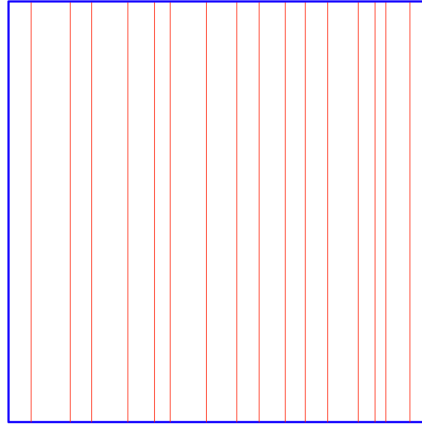
Intuitively, we are zooming into the measure at p by scaling space by r and then compensating by scaling the measure to reverse the scaling of the measure induced by the dilation of space. The α that gives us a non-zero and non-infinite result will be the dimension of the measure at that point.



- $\{x \mid d(x, S) = \alpha\} \subset \bigcup_{i=1}^n \{x \mid d(x, S_i) = \alpha\}$
- $d(p, S_k) = \alpha$
- $\{x \mid d(x, S_k) = \alpha\} \cap B(p, \alpha) \subset C_k(p)$
- $C_k(p)$ is continuous in p !

$\Rightarrow \{x \mid d(x, S_k) = \alpha\} = \text{locally Lipschitz}$
 $= \text{rectifiable} !$

Figure 32: The key construction in proving that level sets of distance functions are rectifiable.



- $\{q_i\}_{i=1}^{\infty} = \mathbb{Q} \cap [0,1]$
- $L_i \equiv \{(x,y) \mid x=q_i, 0 \leq y \leq 1\}$

- $\mu \equiv \sum_{i=1}^{\infty} \frac{1}{2^i} \mathcal{H}^1 \llcorner L_i$



i) $\mu([0,1]^2) = 1$

ii) $\text{supp}(\mu) = [0,1]^2$

Figure 33: The measure μ uses non-integral densities to tame a rectifiable set that is not locally finite.

5.7.5 Local Approximation and Densities

Coming back to one of the motivating questions at the beginning of this section, we encountered the reduced boundary. In order to discuss this, we need the notion of the density of a set.

Figure 34 illustrates the fact that the 2 dimensional density of a set at a point on the topological boundary of a set need not be $\frac{1}{2}$. Defining

$$\theta_*^k(E, x) \equiv \liminf_{r \rightarrow 0} \frac{\mathcal{H}^k(E \cap B(x, r))}{\omega(k)r^k}$$

$$\theta^{k,*}(E, x) \equiv \limsup_{r \rightarrow 0} \frac{\mathcal{H}^k(E \cap B(x, r))}{\omega(k)r^k}$$

and

$$\theta^k(E, x) \equiv \lim_{r \rightarrow 0} \frac{\mathcal{H}^k(E \cap B(x, r))}{\omega(k)r^k}$$

when this limit exists, we have that one of the conditions to be a point in the reduced boundary of a set $E \subset \mathbb{R}^k$ is for $\theta^k(E, x) = \frac{1}{2}$. We define the measure theoretic boundary of a set $E \subset \mathbb{R}^k$ to be the points where both

$$\theta^{k,*}(E, x) > 0$$

and

$$\theta^{k,*}(E^c, x) > 0$$

are true. Here, E^c is the complement of E , i.e. $\mathbb{R}^k \setminus E$

Remark 5.7. *Reminder: I use $\mathcal{H}^k$ even when $\mathcal{L}^k$ makes sense, because, when Lebesgue measure makes sense, the Hausdorff measure and the Lebesgue measure are identical.*

5.7.6 Many Different Boundaries

Mentioned in the opening of this section, the reduced boundary is something that arises in the study of sets of finite perimeter. Figure 35 illustrates the different notions of boundary on a subset of $\mathbb{R}^2$.

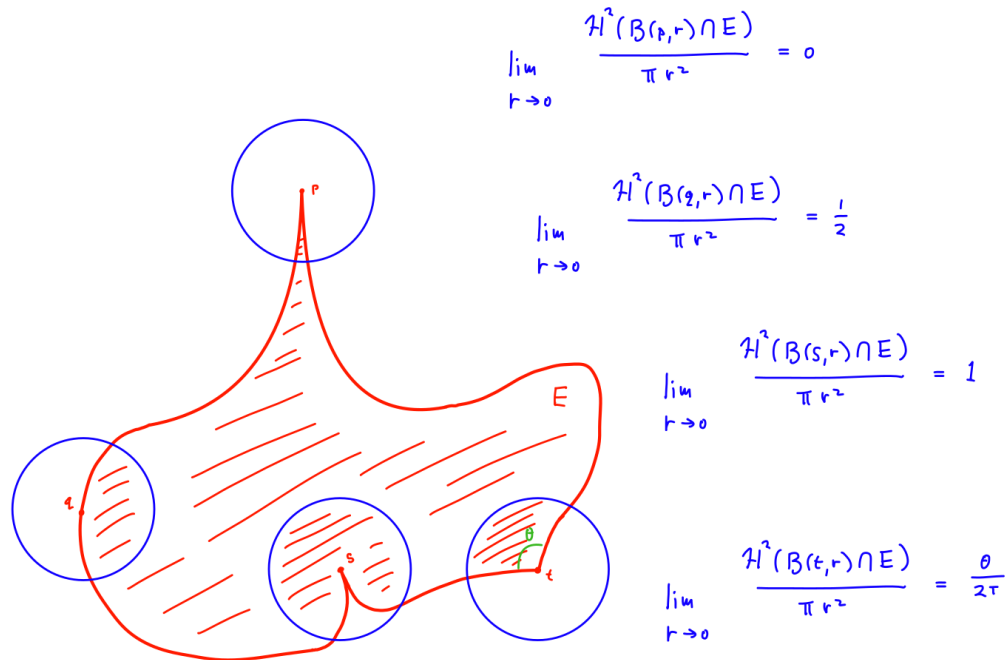


Figure 34: Densities illustrated with a set in $\mathbb{R}^2$

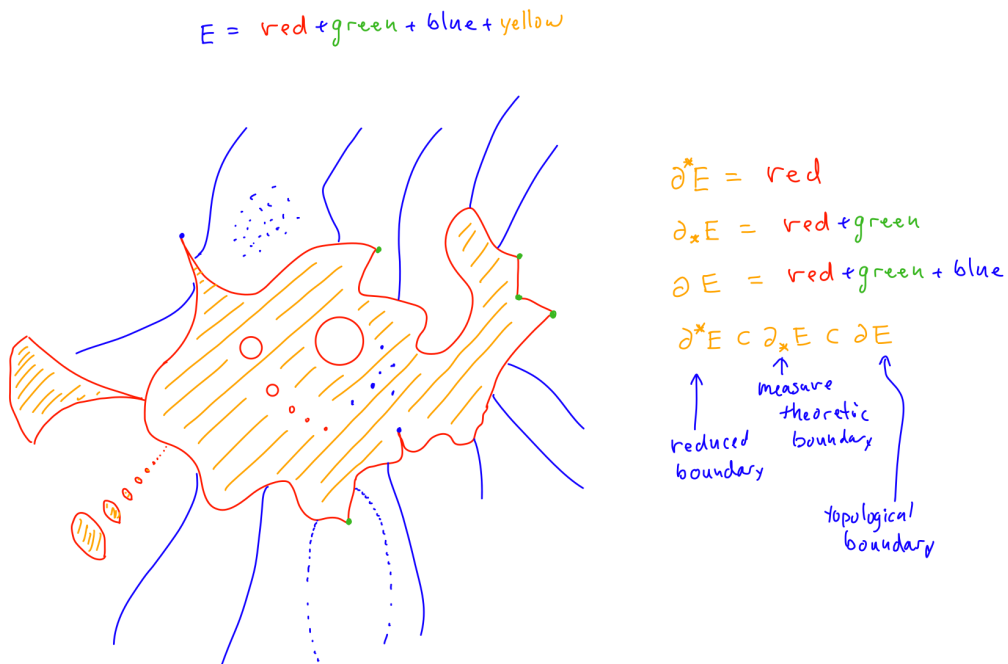


Figure 35: Topological, measure theoretic and reduced boundaries for a set in $\mathbb{R}^2$.

6 Integrating the Gaussian Curvature: Gauss-Bonnet

We now put together the alternative version of the area formula from Section 3.5, the determinant for normal flow in Section 2.2 and what we learned about the degree of a map in Section 4 to get a classical theorem in differential geometry for co-dimension-1 submanifolds of $\mathbb{R}^n$.

6.1 Smoothly Homotopic $n - 1$ -Submanifolds

All the previous work can be collected and turned into a proof of a Gauss-Bonnet theorem for co-dimension 1 submanifolds in $\mathbb{R}^n$.

First there is the determinant of the Gauss map from Section 2.4, recalled here again

Suppose that M is an $n-1$ dimensional manifold in $\mathbb{R}^n$. Then the determinant of the derivative of the Gauss map $G : M \rightarrow \mathcal{S}^{n-1}$ at x is given by

$$\det(D_x G) = \prod_{i=1}^{n-1} \kappa_i(x)$$

where the $\kappa_i(x)$ are the principal curvatures at x . This is also known as the Gauss curvature of M at x .

Next, in Section 4.4.3 we looked closely at the fact that the degree of maps between manifolds are invariant under smooth homotopies concluding that

Degree Invariance for maps $f : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$

Suppose that $H : \Omega \times [0, 1] \rightarrow \mathbb{R}^n$ is a smooth homotopy between the maps $g : \Omega \rightarrow \mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}^n$, i.e. $H(\cdot, 0) = g$ and $H(\cdot, 1) = f$. Then, for all y in the range space $\mathbb{R}^n$ such that $y \notin H(\partial\Omega, t)$ for any $t \in [0, 1]$, we have that

$$\deg(H(\cdot, t), \Omega, y) \text{ does not depend on } t$$

(Here we have assume that we have extended the degree to values that are not regular as suggested at the end of section 4.3.3 so that the degree is defined for all t on all of the complement of $\cup_{t \in [0, 1]} H(\partial\Omega, t)$)

(The subsection 4.4.5 illustrates this for degree of the Gauss map for 1-manifolds smoothly homotopic to the 1-sphere.)

Remark 6.1. *We note that the invariance of degree under homotopy holds for n -manifolds and if they do not have boundaries, then we know that the degree is invariant under the homotopy at all points in the range manifold!*

Next we recalled, in Section 5.1 the version of the area formula we use here is

Mapping Degree Version of the Area formula

$$\int_{\Omega} \det(D_x f) dx = \int_{f(\Omega)} \deg(f, \Omega, y) dy$$

where $\Omega \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz.

Putting all this together we get

Gauss-Bonnet Theorem

If M and N are smooth $n - 1$ -submanifolds in $\mathbb{R}^n$ and M is smoothly homotopic to N , then

$$\int_M \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x) = \int_N \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x)$$

and in particular, if N is smoothly homotopic to the $n - 1$ -sphere in $\mathbb{R}^n$, then

$$\int_N \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x) = n\omega(n), \text{ the } n - 1\text{-measure of the unit sphere in } \mathbb{R}^n$$

Proof.

1. Suppose that $H(\cdot, t)$ is a smooth (orientation preserving) homotopy mapping $M \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and that $H(\cdot, 0) = id$ (the identity) and $N = H(M, 1)$ is an embedding of M .
2. Let G be the Gauss map. $G \circ H(\cdot, t)$ is therefore a smooth homotopy mapping M into $\mathbb{S}^{n-1}$ and $D(G \circ H) = DG \circ DH$ and $\det(D(G \circ H)) = \det(DG) \det(DH)$.
3. Because $H(\cdot, 0)$ is the identity, the degree of $\deg(G \circ H(\cdot, 0), M, y) = \deg(G, M, y)$
4. Because H is orientation preserving, we have that

$$\begin{aligned} \deg(G, M, y) &= \deg(G \circ H(\cdot, 0), M, y) \text{ (by previous step)} \\ &= \deg(G \circ H(\cdot, 1), M, y) \text{ (by degree invariance under homotopy)} \\ &= \sum_{x \in (G \circ H(\cdot, 1))^{-1}(y)} \text{sign}(\det(DG_{H(x,1)}) \det(D_x H(\cdot, 1))) \\ &= \sum_{x \in (G \circ H)^{-1}(y) \subset M} \text{sign}(\det(DG_{H(x,1)})) \text{ (because H is orientation preserving)} \\ &= \sum_{w \in (G)^{-1}(y) \subset N} \text{sign}(\det(DG_{H(x,1)})) \text{ (because N is an embedding of M)} \\ &= \deg(G, N, y) \end{aligned}$$

See Figure 36 for a schematic of this step.

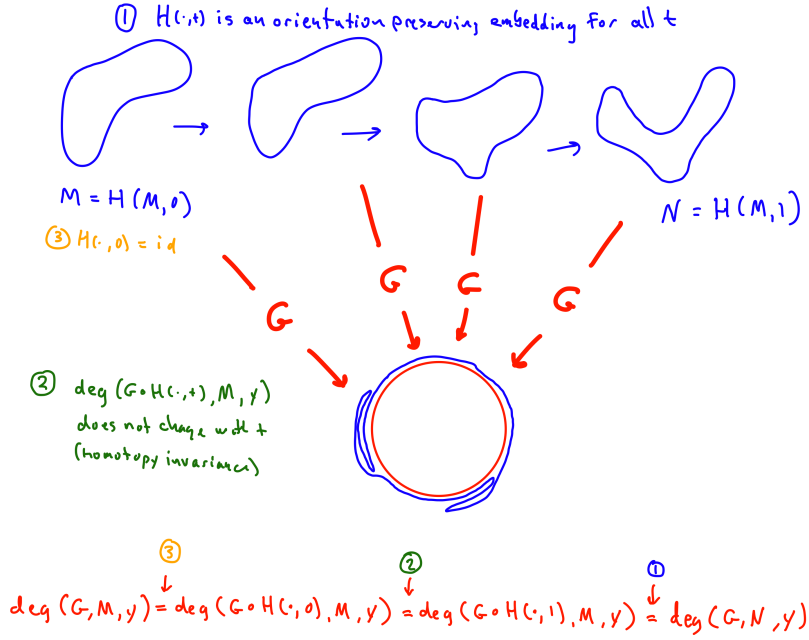


Figure 36:

5. Since the determinant of DG is $\prod_{i=1}^{n-1} \kappa_i$, the previous step in the proof (step 4) plus the alternate version of the area formula above tells us that

$$\begin{aligned}
 \int_M \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x) &= \int_M DG d\mathcal{H}^{n-1} \\
 &= \int_{\mathbb{S}^{n-1}} \deg(G, M, y) d\mathcal{H}^{n-1}(y) \text{ (area formula)} \\
 &= \int_{\mathbb{S}^{n-1}} \deg(G, N, y) d\mathcal{H}^{n-1}(y) \text{ (previous step in proof)} \\
 &= \int_N DG d\mathcal{H}^{n-1}(x) \text{ (area formula)} \\
 &= \int_N \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x)
 \end{aligned}$$

6. Finally, because the Gaussian curvature of the unit sphere is 1 everywhere on the sphere, we get

$$\begin{aligned}
 \int_{\mathbb{S}^{n-1}} \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} 1 \\
 &= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \\
 &= n\omega(n)
 \end{aligned}$$

□

Remark 6.2. Instead of the last step, we could reason as follows ... using what we have already encountered, we know that the Gauss map on the sphere is the identity map of the sphere to the

sphere and therefore has degree 1 everywhere.

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \Pi_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1}(x) &= \int_{\mathbb{S}^{n-1}} \det(D_x G) d\mathcal{H}^{n-1}(x) \\
&= \int_{\mathbb{S}^{n-1}} \deg(G, \mathbb{S}^{n-1}, y) d\mathcal{H}^{n-1}(y) \\
&= \int_{\mathbb{S}^{n-1}} 1 d\mathcal{H}^{n-1}(y) \\
&= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \\
&= n\omega(n)
\end{aligned}$$

and, because the sphere is perfectly symmetric (implying that $\Pi_{i=1}^{n-1} \kappa_i$ is constant), we have calculated – in an unnecessarily and very complicated way – that the Gaussian curvature of the unit sphere is 1!

6.2 Compact 2-dimensional Submanifolds of $\mathbb{R}^3$

The previous section lets us know that the integral of the Gaussian curvature over a manifold is known if this is known for a single member of the equivalence class of submanifolds that are homotopically equivalent. We used that to get that the integral of the Gaussian curvature over manifolds smoothly equivalent to the $n - 1$ sphere is $n\omega(n)$.

In this section we show that for compact 2-manifolds, that integral is always is one of the numbers $\{4\pi, 0 - 4\pi, -8\pi, -12\pi, \dots\}$.

Using two facts,

1. homotopically equivalent co-dimension 1 submanifolds have the same integral of Gaussian curvature (the theorem we just proved), and
2. the only orientable 2-manifolds in $\mathbb{R}^3$ are the 2-sphere, the torus, the two hole “torus”, the three hole “torus”, etc.,

we need only compute that integral for the sphere (we already did that above!) and an n-hole torus (or, if you prefer, a sphere with n-handles).

To do this we begin with the usual 1-hole torus. See Figure 37. We can partition the torus into two pieces (The blue and red pieces at the bottom of Figure 37) and the Gauss map G maps each of these onto the unit sphere (Verify this!⁹) Now convince yourself that on the inside part (red piece), the orientation is reversed by the Gauss map.

Because each point of the range of G (the unit sphere) comes from two points on the torus (except for a set with 2-dimensional measure zero!) and one of those points comes from the outside piece and one from the inside piece (the blue and red pieces at the bottom of Figure 37), we have one point in $G^{-1}(y)$ where DG is positive and one point in $G^{-1}(y)$ where DG is negative, so the degree is zero on the range of G , $\mathbb{S}^2$. Thus, the integral of the Gaussian curvature on the torus is 0. Once we

⁹It is true that it the top horizontal circle and bottom horizontal circles are all in one or the other of the blue and red pieces, then the normals pointing straight up and down are in the range of just one of the pieces under G . But this is a 0-measure issue and therefore effects nothing we compute.

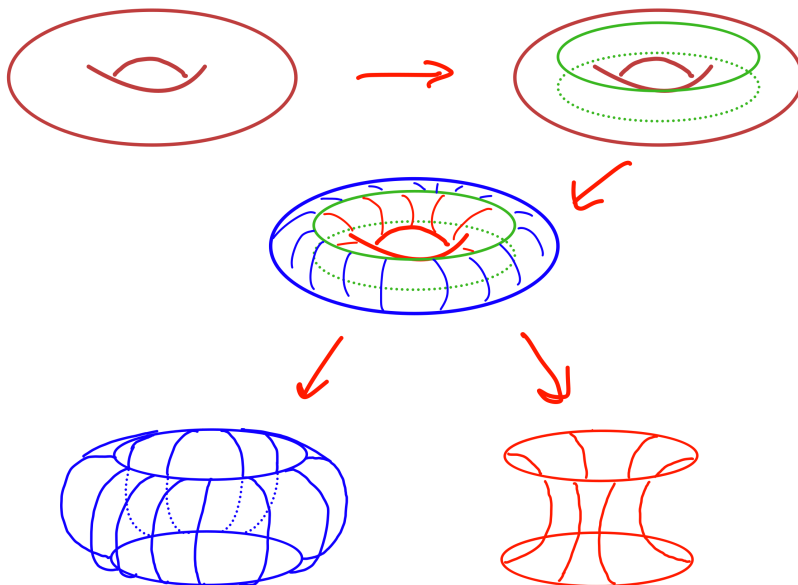


Figure 37: We partition the torus into two pieces that are each mapped onto the entire unit sphere by the Gauss map G . (Actually, denoting the red and blue pieces, respectively, I and O , neither $\mathbb{S}^2 \setminus G(I)$ nor $\mathbb{S}^2 \setminus G(O)$ have more than 2 points in them.)

have convinced ourselves that the degree of the Gauss map on the torus is 0 everywhere, working out that the Gauss map on the 3-hole torus shown in Figure 38 has degree -2 , is relatively simple. It is then clear that for a torus with N holes, $N = 0, 1, 2, 3, \dots$, the degree is $1 - N$ and the resulting integral of the Gaussian curvature is $4\pi(1 - N)$. This yields the set of possible Gaussian curvature integrals promised above $\dots 4\pi \cdot \{1, 0, -1, -2, -3, \dots\}$.

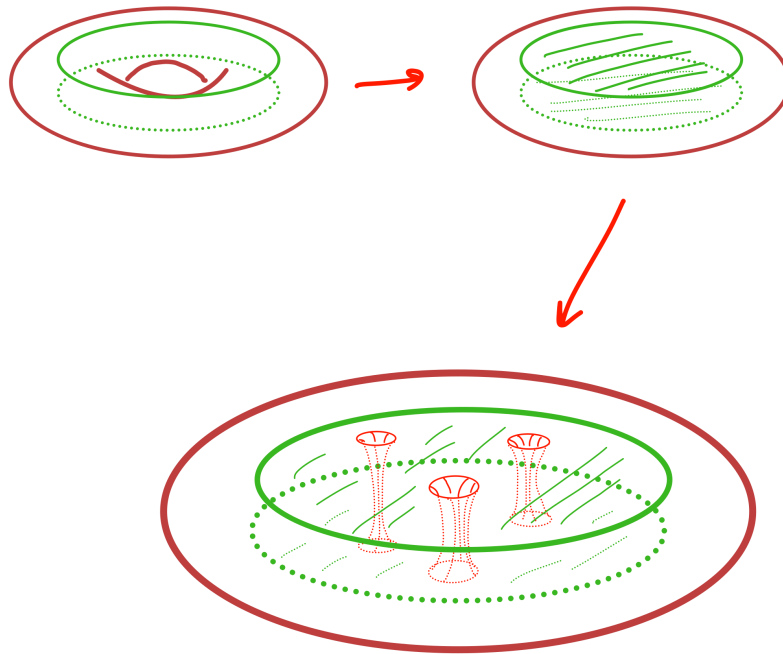


Figure 38: Use what we discussed just above, for the usual torus with one hole, to convince yourself the degree of the Gauss map G on this 3-hole torus is exactly -2. The construction suggested by the figure is to cut the torus on the upper horizontal circle and the lower horizontal circle and glue in two disks to get a (topological) sphere. Then cut little holes and insert three skinny, scaled, inside torus pieces ... everything done smoothly.

7 Evolving the Volume: Tube Formulas and Reach

The beautiful tube formula giving the volume swept out by the normal flow of a surface using what we learned in Section 2.2 is now accessible to us using what we have learned up to this point. (These ideas date back to Steiner and Minkowski in the 19th century, yet are still of interest in more general contexts.)

7.1 Positive Reach

In 1959, in a paper titled “Curvature Measures” Federer introduced the notion of the *reach* of a set E , $\text{re}(E)$, the supremum of distances from the set E such that every point in E^c whose distance is less than or equal to $\text{re}(E)$ has a unique closest point in E . A bit more succinctly,

$$\text{re}(E) \equiv \sup\{r \mid d(x, E) \leq r \text{ implies there is a unique closest point in } E \text{ to } x\}$$

An intuitively appealing and, simultaneously, precise, equivalent, definition is that the reach of a set is the supremum of the radii of balls that can be rolled around ∂E and never touch more than one point of ∂E . See Figure 39.

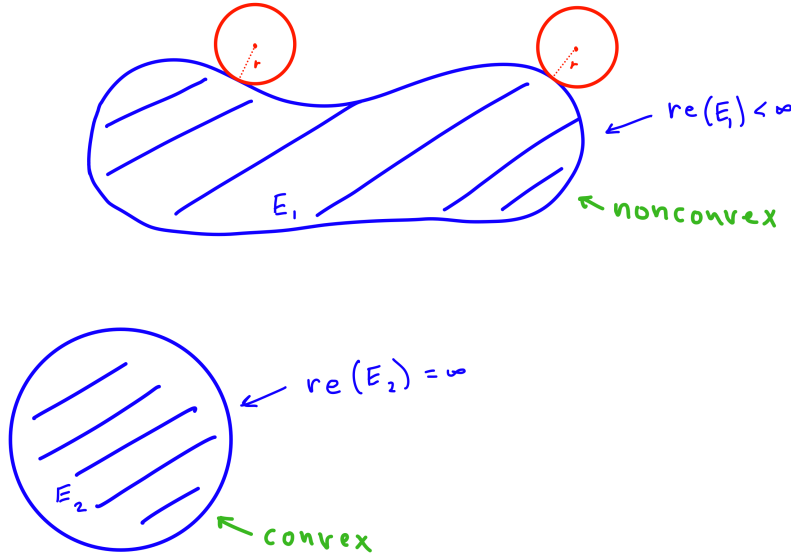


Figure 39: The reach $\text{re}(E_1)$ is bounded by $\frac{1}{\sup_{s \in \partial E_1} |\kappa(s)|}$. In the case of E_1 , the point s that maximizes $|\kappa(s)|$ is found on the top, in the negatively curved portion of the boundary. On the other hand, E_2 is a ball and clearly has infinite reach!

The reach is also the infimum of the distance you must travel along a normal to find an intersection with another, separate normal to some other point. See Figure 40.

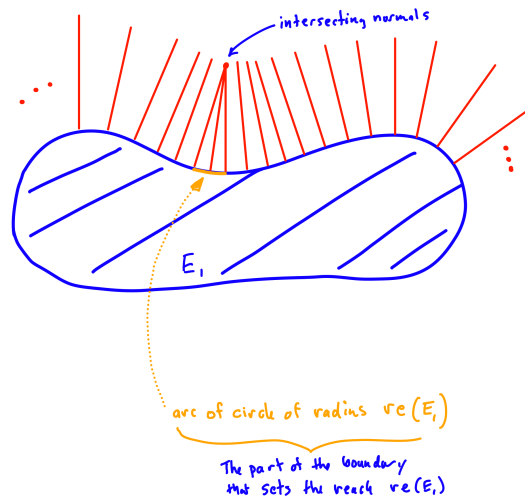


Figure 40: Flowing along the normals, past a distance of $re(E_1)$, introduces singularities into the flow induced map on ∂E_1 . Up until $r = re(E_1)$, the tube formula discussed in the Section 7.2 is valid.

We care about reach because, for a smooth set E , any normal flow that flows out less than the reach of the set yields a surface whose volume is the integral of the expansion factor encountered in Section 2.2. In other words, the tube formula (the subject of the next section), works for any distance $r < \text{re}(E)^a$.

^aActually, the volume of the expanded set $E + rB(0,1)$ is correct for all $r \leq \text{re}(E)$ but the $n - 1$ -measure of the set $\partial(E + rB(0,1))$ is not necessarily valid for $r = \text{re}(E)$

7.2 The Tube Formula for (Smooth) Normal Flow

Previously, using the simple observations about how lengths of curves change as we push out along normals, we calculated how the 2-dimensional measure of a surface changes when we flow it out along normals. (See Sections 2.1 and 2.2.) Figure 41, repeated from Section 2.2, is a reminder of the result that says that any small piece of the surface at the point x , pushed out along the normals by a distance ϵ , experiences a fractional change in area given by the product $(1 + \epsilon\kappa_1(x))(1 + \epsilon\kappa_2(x))$. Now suppose that $\Omega \subset \mathbb{R}^3$ that we expand by $\epsilon < \text{re}(\Omega)$. The change in the area of a boundary as

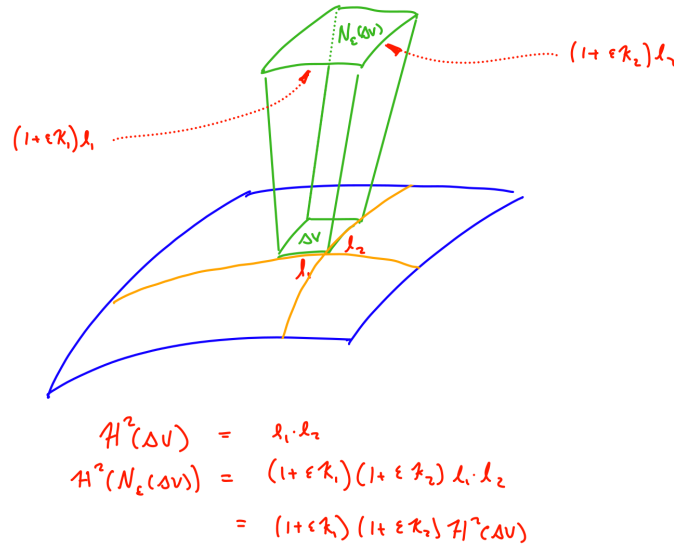


Figure 41: Using the principal curvature directions as a basis, the expansion of the area is given by the product of the dilation in each direction.

we move out along the normals is obtained by simply integrating this product over the boundary:

$$\begin{aligned}
\mathcal{H}^2(\partial(\Omega + B_\epsilon)) &= \int_{\partial\Omega} (1 + \epsilon\kappa_1)(1 + \epsilon\kappa_2) d\mathcal{H}^2 \\
&= \int_{\partial\Omega} 1 d\mathcal{H}^2 + \epsilon \int_{\partial\Omega} (\kappa_1 + \kappa_2) d\mathcal{H}^2 + \epsilon^2 \int_{\partial\Omega} \kappa_1 \kappa_2 d\mathcal{H}^2 \\
&= \mathcal{H}^2(\partial\Omega) + \epsilon \underbrace{\int_{\partial\Omega} (\kappa_1 + \kappa_2) d\mathcal{H}^2}_{\text{Mean Curvature}} + \epsilon^2 \underbrace{\int_{\partial\Omega} \kappa_1 \kappa_2 d\mathcal{H}^2}_{\substack{\text{Gaussian} \\ \text{Curvature}}}
\end{aligned}$$

Using the coarea formula (which, recalling Remark 5.4 in Section 5.7.1, is a nonlinear version of Fubini's Theorem) we get:

$$\begin{aligned}
\mathcal{H}^3(\Omega + B_r) &= \mathcal{H}^3(\Omega) + \int_{\epsilon=0}^{\epsilon=r} \mathcal{H}^2(\partial(\Omega + B_\epsilon)) d\epsilon \\
&= \mathcal{H}^3(\Omega) + r \int_{\partial\Omega} 1 d\mathcal{H}^2 + \frac{r^2}{2} \int_{\partial\Omega} (\kappa_1 + \kappa_2) d\mathcal{H}^2 \\
&\quad + \frac{r^3}{3} \int_{\partial\Omega} \kappa_1 \kappa_2 d\mathcal{H}^2 \\
&= a_0 + a_1 r + a_2 r^2 + a_3 r^3
\end{aligned}$$

where the a_i are integrals of various curvature measures: a_0 = volume of Ω , a_1 = area of the boundary $\partial\Omega$, a_2 = integral of the mean curvature over $\partial\Omega$, and a_3 = the integral of Gauss curvature over $\partial\Omega$. (Of course we just saw, in Section 6, that a_3 is a topological constant.)

The formula for higher dimensions follows from integrating $\prod_{i=1}^{n-1} (1 + \epsilon\kappa_i(x))$ over the boundary $\partial\Omega$ and yields

$$\begin{aligned}
\mathcal{H}^n(\Omega + B_r) &= \mathcal{H}^n(\Omega) + \int_{\epsilon=0}^{\epsilon=r} \mathcal{H}^{n-1}(\partial(\Omega + B_\epsilon)) d\epsilon \\
&= \mathcal{H}^n(\Omega) + r \int_{\partial\Omega} 1 d\mathcal{H}^{n-1} + \frac{r^2}{2} \int_{\partial\Omega} \left(\sum_{i=1}^{n-1} \kappa_i \right) d\mathcal{H}^{n-1} \\
&\quad + \dots + \frac{r^n}{n} \int_{\partial\Omega} \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1} \\
&= a_0 + a_1 r + a_2 r^2 + \dots + a_n r^n
\end{aligned}$$

where the a_i 's are the result of integrating the various curvatures over $\partial\Omega$.

7.3 Generalization to Convex Bodies

For a smooth submanifold E , we know that the normal vectors at a point $p \in E$ are vectors in $T_E^\perp(p)$, the orthogonal complement to $T_E(p)$, the tangent space of E at p . See Figure 42. In the

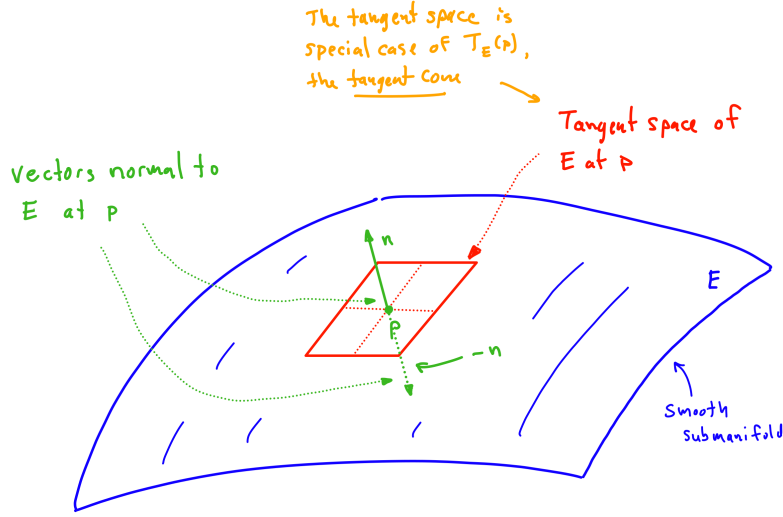


Figure 42: The tangent spaces at points on a smooth manifold are special cases of tangent cones.

case that E is not smooth, the “normal vector” to E (or ∂E) at a point p turns into a normal cone that can range from a single point (the 0 vector) to a subspace to an entire half space. To understand this, we must first understand *tangent cones*, a generalization of the notion of tangent space.

We say that C is the cone generated by E if, to get C we simply take the union of all the rays from the origin that pass through some point in E . We say C is the closed cone generated by E if C is the closure of the union of all the rays from the origin that pass through some point in E . Now one version of the definition of tangent cone.

$T_E(p)$, the tangent cone of E at p is set of rays from the origin that are, for all ϵ , in the closed cone generated by the set of directions $\frac{x-p}{|x-p|}$ obtained from points $x \in (B(p, \epsilon) \setminus \{p\}) \cap E$.

Much more succinctly, defining $\mathbb{R}^+$ to be the set of real numbers in $[0, \infty)$,

$$T_E(p) \equiv \mathbb{R}^+ \cdot \left(\bigcap_n \text{clos} \left\{ \frac{x-p}{|x-p|} : x \in (B(p, \frac{1}{n}) \setminus \{p\}) \cap E \right\} \right)$$

Now we can define the normal cones very simply.

$N_E(p)$, the normal cone to E at p is the set of vectors $\{n : \langle n, v \rangle \leq 0 \ \forall v \in T_E(p)\}$

At this point it will be helpful to study Figures 43-46 which illustrates what we have said so far in this section.

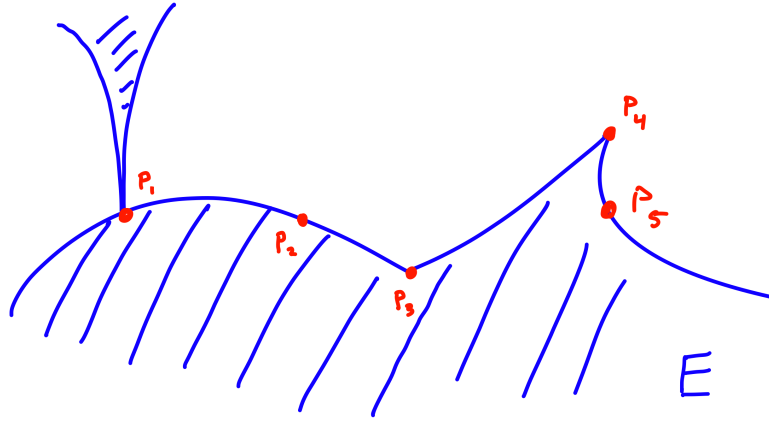


Figure 43: A two dimensional set E whose tangent and normal cones will be explored.

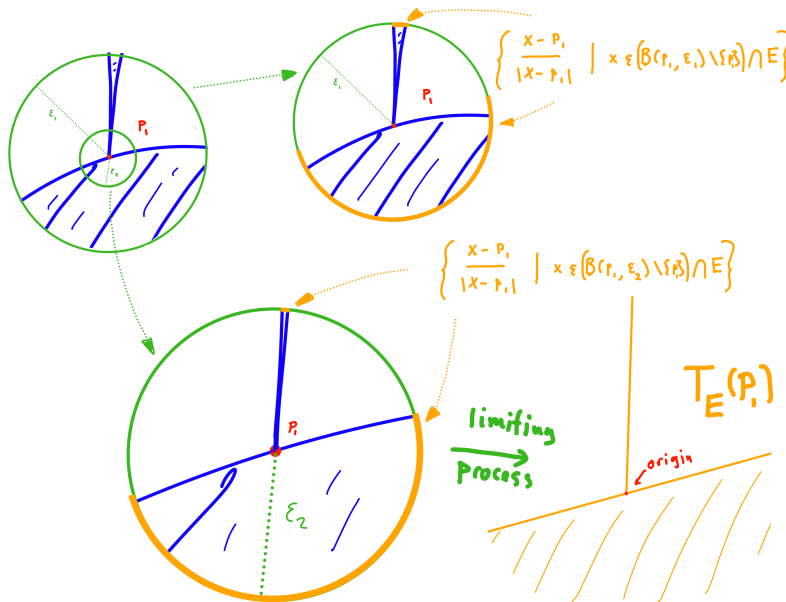


Figure 44: The construction of the limiting process in the definition of tangent cone $T_E(p)$.

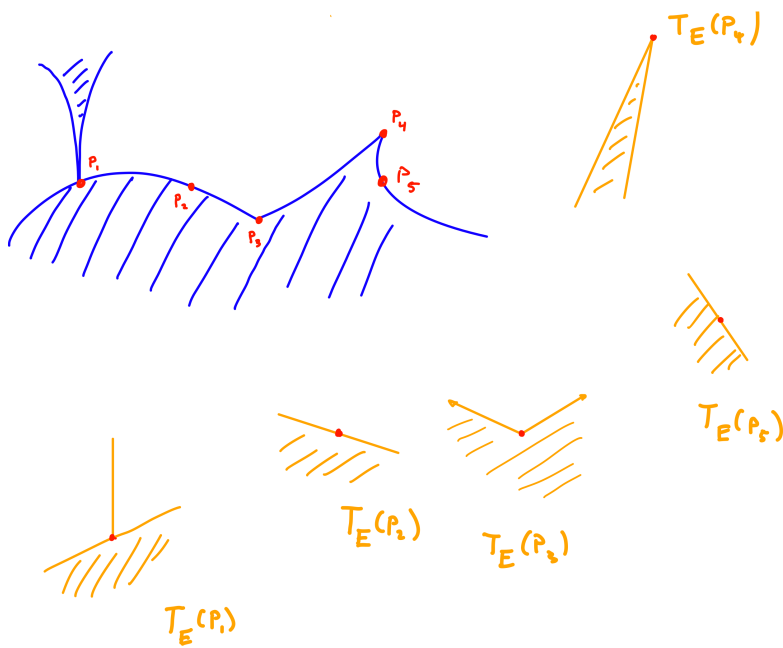


Figure 45: The tangent cones at $p_1 - p_5$.

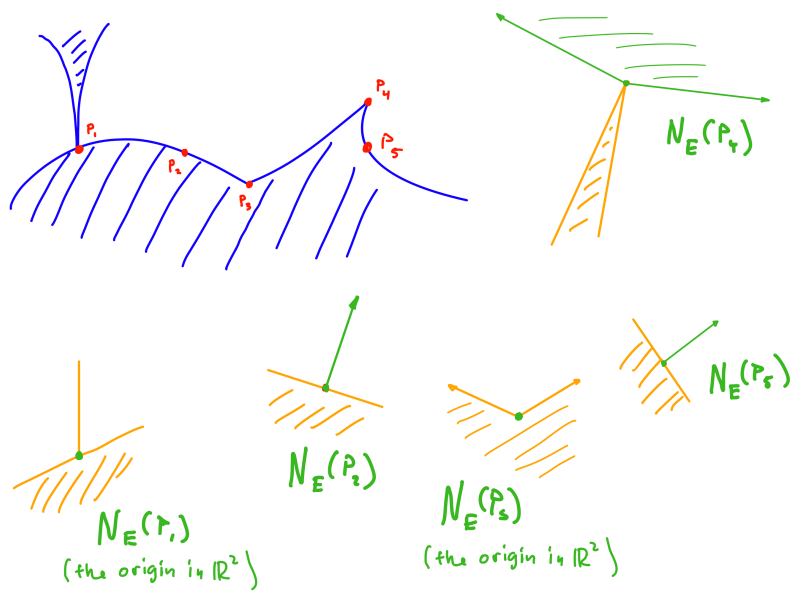


Figure 46: The tangent cones and normal cones at $p_1 - p_5$.

While there are a huge number of generalization directions we could take, I decided that a close look at the solid square in $\mathbb{R}^2$ and the solid cube in $\mathbb{R}^3$ would be a good invitation to further exploration.

The basic idea for convex sets is that because convex sets need not be smooth, they can have normal cones which consist of more than one direction. This leads to the curvatures (the things we integrate to get volumes of tubes) needing to be generalized to include measures.

To help us look at this a bit more carefully, I give the formula for the smooth case in which we are interested in the area generated by a piece of the boundary of the set Ω .

Let $N_\epsilon^{\partial\Omega}(E)$ denote the flow along the normals, for a distance of ϵ , of the set $E \subset \partial\Omega$. We can calculate the $n - 1$ -measure of this set by integrating $\prod_{i=1}^{n-1} (1 + \epsilon \kappa_i(x))$ over $E \subset \partial\Omega$ as follows

$$\begin{aligned} \mathcal{H}^{n-1}(N_\epsilon^{\partial\Omega}(E)) &= \int_{E \subset \partial\Omega} \prod_{i=1}^{n-1} (1 + \epsilon \kappa_i) d\mathcal{H}^{n-1} \\ &= \int_{E \subset \partial\Omega} 1 d\mathcal{H}^{n-1} + \epsilon \int_{E \subset \partial\Omega} \sum_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1} + \epsilon^2 \int_{E \subset \partial\Omega} \sum_{i \neq j} \kappa_i \kappa_j d\mathcal{H}^{n-1} + \dots \\ &\quad \dots + \epsilon^{n-1} \int_{E \subset \partial\Omega} \prod_{i=1}^{n-1} \kappa_i d\mathcal{H}^{n-1} \end{aligned}$$

Thinking about this formula for nonsmooth convex sets Ω helps us get the correct generalization.

To avoid the even deeper waters that would lengthen this beyond the initial look I am giving here, the approach I take here will be to simply present the measures that replace the usual curvatures and then discuss the results, leaving the door open for the reader to pose and try to answer questions that come up.

7.3.1 The Square

We begin with the unit square (see Figure 47). In that figure, the set we are expanding is the blue square and the formula for the length of the boundary of the expanded S is given by

$$\mathcal{H}^1(\partial(S + \epsilon B)) = \int_{\partial S} 1 d\mathcal{H}^1(x) + \epsilon \int_{\partial S} d\mu(x)$$

where $B = B(0, 1)$ and

$$\begin{aligned} \mu &= \sum_{c \in \text{corners of } S} \frac{\pi}{2} \delta(x - c) \\ &= \sum_{c \in \text{corners of } S} \frac{\pi}{2} \mathcal{H}^0 \llcorner \{c\} \end{aligned}$$

So, in this case the mean curvature weighted $\mathcal{H}^1$ measure, $\kappa \llcorner \mathcal{H}^1 = \kappa d\mathcal{H}^1$ is replaced by the measure given by four point masses at each of the corners, each weighted by $\frac{\pi}{2}$.

If we are just interested in $N_\epsilon^{\partial S}(F)$ for some $F \subset \partial S$, we have instead

$$\mathcal{H}^1(N_\epsilon^{\partial S}(F)) = \int_F 1 d\mathcal{H}^1(x) + \epsilon \int_F d\mu(x).$$

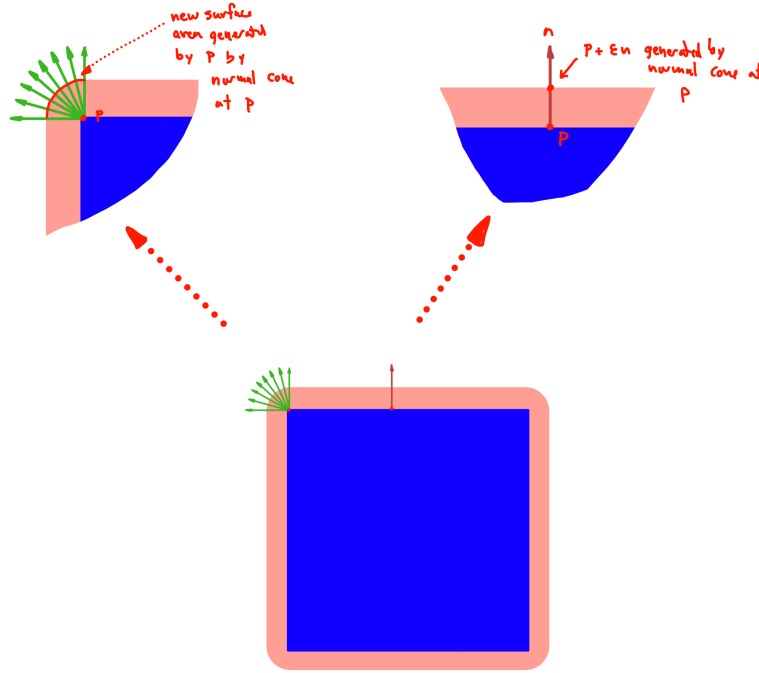


Figure 47: The boundary generated by flowing out along the normal cones a distance of ϵ now calls for measures that are singular with respect to $\mathcal{H}^1$. In particular, we need point masses (Dirac delta “functions”) at each of the corners of the blue square S . Instead of the curvature being distributed smoothly across ∂S , the boundary of S , it is concentrated at four points.

Exercise: develop, using the square example for inspiration, the correct formula for any given convex polygon in $\mathbb{R}^2$. What about the case in which there are a countably infinite number of corners in your polygon?

7.3.2 The Cube

In Figures 48 and 49, the set we are expanding is the solid unit cube and the formula for the area of the boundary of the expanded C is given by

$$\mathcal{H}^2(\partial(C + \epsilon B)) = \int_{\partial C} 1 d\mathcal{H}^1(x) + \epsilon \int_{\partial C} d\mu_1(x) + \epsilon^2 \int_{\partial C} d\mu_2(x)$$

where again $B = B(0, 1)$,

$$\mu_1 = \frac{\pi}{2} \mathcal{H}^1 \llcorner (\text{Edge set of } C)$$

and

$$\begin{aligned}\mu_2 &= \sum_{c \in (\text{corners of } C)} \frac{4\pi}{8} \delta(x - c) \\ &= \sum_{c \in (\text{corners of } C)} \frac{4\pi}{8} \mathcal{H}^0 \llcorner \{c\}.\end{aligned}$$

If we are just interested in $N_\epsilon^{\partial C}(F)$ for some $F \subset \partial C$, we have instead

$$\mathcal{H}^1(N_\epsilon^{\partial C}(F)) = \int_F 1 d\mathcal{H}^1(x) + \epsilon \int_F d\mu_1(x) + \epsilon^2 \int_F d\mu_2(x).$$

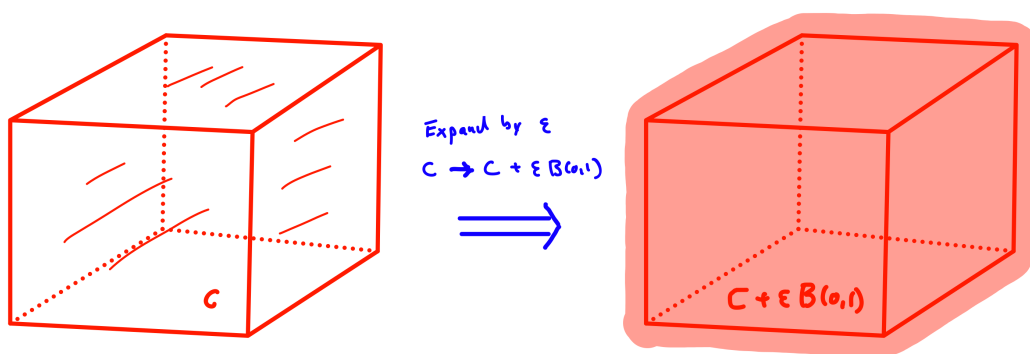


Figure 48: The cube, expanded by ϵ .

Exercise: Use what is shown in Figure 49 to justify for yourself the two formulas given for the cube area expansions. What is the reach for the unit square and unit cube? If P is some convex 3-dimensional polygon in $\mathbb{R}^3$ with N sides, what do you need to know about the polygon to write down the the analogous curvature measures μ_1 (involving a weighted $\mathcal{H}^1$ measure) and μ_2 (involving a weighted $\mathcal{H}^0$ measure)?

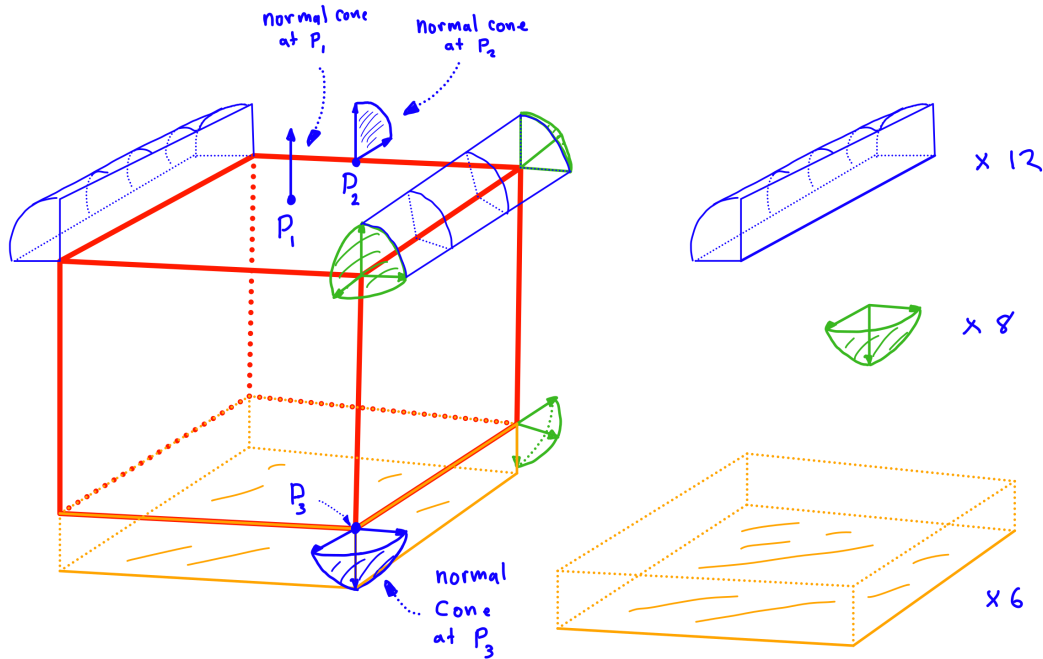


Figure 49: The details of the boundary and volume expansions. This should help with the justification for the formulas given just above.

8 Comments and References for Further Study

I begin by will reiterating what I said at the beginning.

These notes are intended to spark an interest in further study of the topics exposed and begin by seeing the connections between pieces not always explained in such a way as to make the connections clear. I have written an informal exposition that will undoubtedly leave the more precise souls out there uneasy. The solution for them is easy – look at the references I highlight in this section and the bibliography. All of them are at least a bit more precise.

For the rest of the readers, I claim that an informal introduction such as these notes (which are absolutely much better presented by someone in a lecture discussion formal – I recommend reading these notes with someone like that at hand!) will help illuminate pieces of non-linear analysis that are often not as well motivated or illustrated as they could be.

Here are notes on references I recommend for deeper studies.

Milnor *Topology from the Differentiable Viewpoint* [15] is a beautiful little book (64 pages) containing a rich set of explanations of ideas in differential topology. I recommend it very highly.

Deimling I have used *Nonlinear Functional Analysis* [4] for a detailed development of Brouwer degree. It is definitely worth having as a reference. It used to be available as a paperback from Dover but it appears that Springer (the original publisher) has started printing it again. You can still purchase the e-copy from Dover for 19\$.

Needham *Visual Differential Geometry and Forms* [17] is an unusually beautiful book on geom-

etry. I recommend it very, very strongly. Needham uses intuitions and visual exposition extraordinarily well.

Morgan Frank Morgan's beautiful *Geometric Measure Theory: A Beginners Guide* [16] is the best first look at geometric measure theory. If you read the first three chapters, you will get a very good, fairly fast introduction to rectifiability. The whole book is worth reading of course, but the first 5 chapters is a fast introduction to the geometric measure theory approach to minimal surfaces created by Federer and Fleming. I traditionally require my students to study (at least) the first five chapters, in order to get a good, intuitive grasp of the subject. The book is intended as an introduction to Federer's tome.

Evans and Gariepy I learned advanced measure theory from Evans and Gariepy's *Measure Theory and Fine Properties of Functions* [5] and I recommend it very highly for all the things they cover. While they do cover sets of finite perimeter in detail (and this is one very effective way to approach important examples of co-dimension-1 rectifiable sets, they do not go into the theory of rectifiable sets that are not boundaries. But they cover a large amount of relevant material.

Stein Elias Stein was a master teacher and superb mathematician working in harmonic analysis. His *Singular Integrals and Differentiability Properties of Functions* [19] is a classic and a model of clear, inspiring exposition. I think that everyone should at least read the first chapter to see both the contents of that chapter and find out if they feel compelled to read/study the rest of the book.

Mattila *Geometry of Sets and Measures in Euclidean Space* [14] is the standard reference on the rectifiable sets and measures approach (as opposed to the currents approach) to geometric measure theory and the main place harmonic analysis folk learn geometric measure theory. I have taught from the book and find it well written and very useful.

Falconer *The Geometry of Fractal Sets* [6] is a very nice, short (and advanced) book packed with information on rectifiability and fractals. I recommend it strongly.

Simon I think Leon Simon's *Introduction to Geometric Measure Theory* [18] is now my favorite more advanced book on geometric measure theory. You should get the 2018 revision.

Tu I recently taught from both *An Introduction to Manifolds* and *Differential Geometry: Connections, Curvature and Characteristic Classes* ([20], [21]) and find that they are very well written (and would also work well for independent study). In the end, I may slightly prefer Boothby's book (also commented on here) but I think that these two books should be owned by anyone who wants to master the basics of differential geometry.

Guillemin and Pollack *Differential Topology* [9]. There is a sense in which this book is an unfolding of Milnor's little book *Topology from the Differentiable Viewpoint*, something Guillemin and Pollack even admit to in the preface. Even though I have not taught from the book yet, every time I look at it I think that doing that would be a fantastic idea!

Kesavan I like some pieces of *Nonlinear Functional Analysis* [10] and other pieces, not so much. It is inexpensive and short and also perhaps a bit clearer in presentation than average. I do recommend having it on hand because it is priced well and for some topics, it is quite good.

Avez I learned higher dimensional calculus (calculus in Banach spaces) first from *Differential Calculus*[1] under the influence of John Erdman at Portland State University. John was an extraordinary teacher who taught with the Modified Moore Method that ambitious students found inspiring. Avez’ recommendation (seen as I was writing this entry!) is also the reason I finally (in the last 30 minutes!) took L.C. Young’s book (also, now, in this list of books that I comment on) off the shelf. That alone would be reason enough to stick Avez in this list! The edition of Avez cited here is the 2020 Dover republication of the first English language 1986 edition that I learned from in the 1990’s.

Young L.C. Young’s 1980 *Lectures on the Calculus of Variations and Optimal Control Theory* [22] is a gem (as implied by my short notes in Avez). I have read only a small part of the book, but that is enough to absolutely guarantee I will read the rest. He was Wendell Fleming’s advisor at University of Wisconsin and Wendell was Bill Allard’s advisor at Brown university ... and Bill is one of 2 primary people from whom I learned pieces of geometric measure theory, (in addition to lots of self-study).

Lee *Introduction to Topological Manifolds*, *Introduction to Smooth Manifolds*, and *Riemannian Manifolds: an Introduction to Curvature* [12], [13], and [11] are a trinity of books by an exceptional expositor. Every student of differential geometry and topology must own all three of these books. The only criticism is also a strength – sometimes the explanations are a bit too long because they contain so much detail.

Boothby *Introduction to Differential Manifolds and Riemannian Geometry* [2] I think this is still my favorite text on differential geometry, though I think that the fact it was the first text I studied figures into that estimation. At any rate, while it is not perfect, and I would not recommend studying the subject without Lee’s books or Tu’s books or Needham’s book, I still recommend this book strongly. Something I can say in it’s favor is that all the other options recommended here take longer to cover the same topics.

De Lellis *Rectifiable Sets, Densities and Tangent Measures* [3] is my favorite book on things rectifiable even though it is an exposition aimed somewhat narrowly at David Preiss’ masterful 1987 *Geometry of Measures in $\mathbb{R}^n$: distribution, rectifiability and densities* and does not cover enough to claim it is a complete course on rectifiable sets and measures. On the other hand, there is something special about it and the paper it covers. Federer was reputed to have said he thought Preiss’ paper was the most important mathematical paper – “in geometric analysis” was implied, I suspect – of the 20th century.

Federer For the material in Section 7, the 1959 paper *Curvature Measures* [7] by Herbert Federer is the definitive source. Highly recommended. Not an easy read.

Federer *Geometric Measure Theory* [8] is still the bible on much of geometric measure theory. It is recommended as a reference. It is not easy to read. Bill Allard was at Brown University while Federer was writing the book and read the entire thing, commenting on many pieces, all while he was a graduate student. Those that know the book cannot help but be impressed by this.

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References

- [1] A Avez. *Differential calculus*. Dover Publications, Inc, Mineola, New York, 2020.
- [2] William M. Boothby. *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, 1975.
- [3] Camillo De Lellis. *Rectifiable sets, densities and tangent measures*. European Mathematical Society, 2008.
- [4] Klaus Deimling. *Nonlinear functional analysis*, volume 478. Springer-Verlag Berlin, 1985. This was republished by Dover but Springer is printing it again. You can still get an inexpensive e-copy from dover.
- [5] Lawrence C Evans and Ronald F Gariepy. *Measure theory and fine properties of functions*. Chapman and Hall/CRC, 2nd edition, 2025.
- [6] Kenneth J Falconer. *The geometry of fractal sets*, volume 85. Cambridge university press, 1986.
- [7] Herbert Federer. Curvature measures. *Transactions of the American Mathematical Society*, 93(3):418–491, 1959.
- [8] Herbert Federer. *Geometric Measure Theory.-Reprint of the 1969 Edition*. Springer, 1996.
- [9] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, 1974. ISBN 0-13-212605-2.
- [10] S. Kesavan. *Nonlinear functional analysis : a first course*. Texts and Readings in Mathematics ; v.28. Springer, Singapore, 2nd ed. edition, 2022.
- [11] John M. Lee. *Riemannian Manifolds: An Introduction to Curvature*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, 1997. ISBN 0-387-98322-8.
- [12] John M. Lee. *Introduction to Topological Manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer-Verlag, 2000. ISBN 0-387-95026-5.
- [13] John M. Lee. *Introduction to Smooth manifolds*. Springer, 2012.
- [14] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*, volume 44. Cambridge University Press, 1999.
- [15] John W. Milnor. *Topology From The Differentiable Viewpoint*. University Press of Virginia, Charlottesville, 1965. Eighth printing, 1990.
- [16] Frank Morgan. *Geometric Measure Theory: A Beginner's Guide*. Academic Press, 5th edition, 2016.

- [17] Tristan Needham. *Visual Differential Geometry and Forms: A Mathematical Drama in Five Acts*. Princeton University Press, 2021.
- [18] Leon Simon. *Introduction to Geometric Measure Theory*. 2018 Revision of 1984 lectures, 2018.
- [19] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 1970.
- [20] Loring W. Tu. *An Introduction to Manifolds*. Universitext. Springer New York, New York, NY, 2nd ed. 2011. edition, 2011.
- [21] Loring W. Tu. *Differential Geometry : Connections, Curvature, and Characteristic Classes*. Graduate Texts in Mathematics, 275. Springer International Publishing, New York, NY, 1st ed. 2017. edition, 2017.
- [22] L. C. Young. Lectures on the calculus of variations and optimal control theory, 1980.