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Questions and Answers about Area-Minimizing Surfaces and Geometric Measure Theory

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"Find the surface of smallest area spanning a given contour."

This celebrated problem can bring to mind visions of irridescent soap films or dazzling computer renderings of minimal surfaces. It has been and continues to be the fountainhead of an astonishingly large amount of theoretical and computational mathematics. Indeed, making mathematics of this and more general least area problems opens a virtual Pandora's box of questions: What is a surface? What is the "area" of a surface? What does it mean for a surface to span a boundary? What does it mean to "find" a surface? What about surfaces which are minimal for surface energies other than area? What about surfaces in manifolds or surfaces which are only locally area minimizing? The problem of least area is thus not a single problem but a collection of problems depending on the way one answers these questions. The study of such problems has been one of the central themes of geometric measure theory. Here are informal personal answers to questions which are sometimes asked about area minimization or geometric measure theory.

What is "Plateau's problem"? Plateau refers to Joseph A. Plateau (1801–1883) who is best known for his volumes, *Statique Experimentale et Théoretiques des Liquides Soumis aux Seules Forces Moleculaires* **[Pl]**. With an image of a soap film within a wire frame in mind, Plateau's problem (or the Plateau problem or the problem of Plateau) is the mathematical problem of finding a surface of least area spanning a given curve. The term is also used for a variety of problems about area-minimizing surfaces in general dimensions and codimensions.

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What is an area-minimizing surface? I feel that a surface should be called "area minimizing" if it locally minimizes a reasonable notion of surface area among reasonable competitors. The surfaces in competition may be constrained by requirements such as spanning a boundary, lying in a submanifold, or surrounding given volumes. Examples would then include the following:

- mathematical models for soap films on wire frames and clusters of soap bubbles;
- minimal imbeddings or immersions of a manifold into another manifold;
- level sets of functions such as holomorphic varieties in Kähler manifolds.

Area-minimizing surfaces thus sometimes contain singularities.

What is a mathematical soap film? For me a mathematical soap film can be obtained by the mathematical equivalent of surrounding a boundary wire by a water-filled balloon and then sucking the water out of the balloon, which we assume wishes to minimize its surface area; this models the way a film is formed physically. We are then permitted to make area-decreasing Lipschitz deformations within small regions leaving the boundary wire fixed. Any suitable limiting surface is our soap film. It will be an $(\mathbb{M}, \epsilon, \delta)$ minimal set, as discussed below **[P2]**. F. Kochman describes in **[KF]** how to puncture regions in a soap film in order to obtain a manifold; the procedure works in practice! Other soap films are obtained by surrounding the wire with a torus-shaped balloon, etc.; this is an alternative to puncturing regions.

What is a mathematical soap bubble cluster? By such a cluster I usually mean a solution to the problem of partitioning space into regions of prescribed volumes in such a way as to minimize total interface area.[1] The soap bubble problem has been a benchmark problem for theory, computation, and rendering. The solution will again be an $(\mathbb{M}, \epsilon, \delta)$ minimal set.

What is the area of a surface? Reasonable notions of surface area include the following:

- If the surface is a set, one might mean its Hausdorff measure (sometimes called size) or its Hausdorff measure weighted with an algebraic or topological density (sometimes called mass). Size is minimized by soap films and soap bubbles, while mass is minimized by holomorphic varieties.
- If the surface is represented as a mapping, then one might mean the Hausdorff measure of the image of the mapping, or one might mean the Jacobian integral of the mapping.

What difference does the definition of area make? Different notions of area produce different minimal surface geometries and different kinds of singularities in minimizing surfaces. A smooth $(k-1)$-dimensional submanifold along which three sheets of k-dimensional minimal surface meet at equal angles

[1]This is an example of general partitioning problems discussed in [**A4**, Chapter 6].

of 120° is an example of a singularity which can occur in a k-dimensional size-minimizing surface but will not occur in mass-minimizing surfaces of most kinds.[2] A smooth $(k-2)$-dimensional submanifold around which a smooth k-dimensional minimal surface branches is an example of a singular set which can occur in k-dimensional mass-minimizing surfaces but will not occur in size-minimizing surfaces.

In what ways might an area-minimizing surface might span a boundary?

• If a soap film actually forms on a wire then surely it should "span" the wire in some mathematical sense. One needs to be careful here. J. F. Adams [**R1**, Appendix] showed how to construct a soap film in an unknotted wire which could, mathematically at least, be deformed within itself to the boundary, leaving the boundary fixed throughout. I later discovered that a soap film will form in a wire tied in a loose overhand knot without touching the entire wire; at the time I had been trying to prove that a soap film had to touch all its boundary wire!; note [**P2** and **B3**].

• If the minimal surface is an oriented or unoriented manifold with boundary, then it should span that boundary. Even with this condition there can be surprises as W. P. Thurston and I illustrated in our paper, "Examples of unknotted curves which bound only surfaces of high genus within their convex hulls" [**ATh**].

• Čech homology theory provides another method in which a surface S might span a hole in a boundary C or represent a homology class. One says that an element τ of the Čech homology group $H_{k-1}(C)$ (with suitable coefficients) is spanned by S provided $i_*\tau = 0$ under the inclusion $i: C \xrightarrow{\subset} C \cup S$. This notion was introduced by J. F. Adams in [**R1**, Appendix]. It has been extended substantially by A. T. Fomenko to "spectral" homology as discussed below. In such a context one minimizes size.

• Singular homology theory provides the basis of a final way to describe minimal surface spanning. The integral currents discussed below are modelled by integration of smooth differential forms over Lipschitz singular chains with integer coefficients with boundaries correspondingly determined. The flat chains mod ν are modelled by measure-theoretic Lipschitz singular chains with coefficients in the integers modulo ν. One usually minimizes mass rather than size in this context.

What kinds of area-minimizing surfaces can be computed? The algorithms for surface energy minimization which I know best have been developed by members of the "minimal surface team" of the Geometry Center in Minneapolis. They include the following, all of which are illustrated in the video report *Computing Optimal Geometries* [**T4**].[3]

• Suppose one is given a collection C of oriented boundary curves to be spanned, a surface energy functional (such as surface area) to be minimized, and an acceptable error. J. Sullivan developed an algorithm which will

[2]Such a singularity can occur in mass-minimizing flat chains mod $3k$, however, for positive integers k.

[3]Members of the minimal surface team at the time of the video report were F. Almgren, J. Taylor, K. Brakke, R. Almgren, J. Sullivan, J. Steinke, A. Sufke, O. Holt, and C. Sandvig.

compute an oriented polyhedral surface, spanning C as a current, whose surface energy differs from the infimum by no more that the prescribed error [**Su**]. The combinatorial structure and topological complexity of the surface is part of the output, not the input, of the algorithm. It, however, may not be computationally feasible to implement his algorithm for reasonable errors, like 5%. H. Parks earlier gave a different algorithm for extreme boundary curves [**P1**]. No such algorithm is known if unorientable surfaces are in competition or for two-dimensional surfaces in $\mathbb{R}^4$.

- The "Surface Evolver" is an interactive program for the study of surfaces shaped by surface tension. It is largely the creation of K. Brakke [**B2** and **B4**]. The Evolver program evolves a given initial surface to minimize energy. It can accept and manipulate quite general initial combinatorial structures, to model, for example, soap bubble clusters. It will minimize either area or a crystalline surface energy together with a bulk energy expressed as a surface integral, e.g., gravitational potential energy. During the minimization it will respect volume constraints, boundary constraints, boundary contact angles, and periodicity conditions as required.

A limitation of the Surface Evolver is the requirement that it be given an initial combinatorial structure. During the surface evolution, combinatorial structure can be collapsed but new structure cannot be created, except in the sense of refining triangulations, etc. Even these changes require intervention of the operator.

- The "Voronoi Cell Evolver" is designed to create and evolve complicated combinatorial geometries such as occur in soap bubble clusters, without the requirement of an initial combinatorial input. It is largely the creation of J. Sullivan. In addition to creating soap bubble geometries, one can use the program in the study of grain boundary migration as in an annealing metal. Geometry in the Voronoi Cell Evolver is created and evolved by moving many "control sites" of several different colors, one color for each desired cell of the final soap bubble cluster. The points in the Voronoi cell of each site (i.e., the collection of points in space closer to that site than any other) are given the color of that site. These geometries are manipulated by moving control sites to minimize total interface area while preserving the total volumes of the regions of each color.

- The "Crystalline Surface Creator" of J. Taylor produces a polyhedral disk spanning a given boundary wire which minimizes not surface area but a crystalline surface energy. Based on a theoretical classification of possible dual graphs of an energy minimizer [**T3**], the algorithm generates these candidates and does a quadratic programming minimization over each to select the one of least energy.

What is a good way to display soap bubble geometries? It is a challenge to draw pictures of complicated geometric configurations especially ones with substantial internal structure. One of the newer rendering techniques for static geometries is an algorithm developed by J. Sullivan. It is based on

Fresnel's laws and produces both the colored interference patterns of reflected light and the "Fresnel effect" of decreased transparency at oblique angles. The effects can be striking; see [**AS**]. An alternative to static renderings is computer animations recorded on video tape as in [**T4**].

What is the Geometry Center? "The Geometry Center" is the short title of the National Science and Technology Research Center for Computation and Visualization of Geometric Structures[4], sponsored by the National Science Foundation and the University of Minnesota. This Center grew from the earlier Geometry Supercomputer Project, also in Minneapolis. It is administered by the nineteen member Geometry Group[5] which also serves as its "permanent faculty". The specific mission of the Center includes mathematics and computer science research, software development and distribution, and education, broadly interpreted.

Within the limits of its resources the Center hopes to provide computing and graphics support and assistance to the mathematics and computer science communities.

What can least area theory do for computations? Theoretical insights have sometimes helped create algorithms. The examples I know best are the following:

- Flat duality between currents and forms [**FH3**] led to the algorithms for computing oriented least area surfaces [**PH, Su**] mentioned above.
- Classification of soap bubble and film singularities [**T1**] in part led to the Voronoi methods of minimal surface computation mentioned above.
- Quantization of curvature in dual (Gauss map) representations of crystalline minimal surfaces [**T3**] led to the algorithm for computation of crystalline minimal surfaces mentioned above.

What can computation do for the theory of least area? The obvious answer is that one can use computers to compute examples and to test conjectures. Some parts of the field of minimal surface theory have changed substantially in the past decade because of computers and computer graphics. Well known are the pictures of D. Hoffman and W. Meeks of their discoveries of new global embedded minimal surfaces as discussed and illustrated in [**H1**] and in the video tape [**H2**]. For me personally, the Surface Evolver program makes it easy to compute and render minimal surfaces of many types. This has led to the discovery of new types of soap bubble configurations and to persuasive evidence for the nonexistence of certain fancy bubbles, e.g., a single bubble surrounded by a torus-shaped second bubble.

It is not uncommon in geometric measure theory to prove theorems by compactness arguments. One of the limitations of this method of proof is

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[4]The Geometry Center, 1300 South Second Street, Minneapolis, MN 55415.

[5]Present members of the group are F. Almgren, J. Cannon, B. Chaselle, J. Conway, D. Dobkin, A. Douady, D. Epstein, M. Freedman, P. Hanrahan, J. Hubbard, H. Keynes, B. Mandelbrot, A. Marden (Director), J. Milnor, D. Mumford, C. Peskin, J. Taylor, W. Thurston, A. Wilks.

that one often gets no sense of the real size of constants.[6] According to **[SSY]** there is an a priori bound to the principal curvatures at the origin in an area minimizing surface having boundary on the unit sphere in space. The best numerical estimate so far by that method is 10^6 **[Su]**. My initial instinct was that that estimate is probably off by at least several orders of magnitude. On the other hand a soap film type minimal surface computed by Brakke **[B3]** had to be magnified by 10^6 to reveal fine detail of boundary behavior.

The crystalline variational calculus discussed below, even as a theoretical discipline, seems most naturally expressed in the language of algorithms. Surfaces computed by the Crystal Creator **[A11]** show developing varifold (infinitesimally corrugated) structures and also developing cusps in the boundaries of facets. Based on these examples, J. Taylor has formulated general conjectures about conditions under which such phenomena occur.

Have minimal surface computations been useful outside mathematics? One example which comes to mind is use of the Surface Evolver program by engineer J. Tegart of the Martin-Marietta Corporation in Denver. He uses the program to compute where fuel will be in one of the liquid fuel tanks of the space shuttle in low gravity conditions with various accelerations; fuel position is managed largely by capillary effects of baffles and screens. He reports that computed solutions closely coincide with analytic solutions in those cases where analytic solutions are available **[TJ1]**; see also **[TJ2]**. Also, Hoffman's video **[H1]** shows computer-produced semitransparent projections of solids bounded by certain minimal surfaces which projections strongly resemble transmission electron microscope photographs of certain block copolymers; this suggests that interfaces within the block copolymer are close to being minimal surfaces.

Why are least area problems hard theoretically? Prior to 1960, when one thought of a surface of least area there is a good chance it would have been a mapping $f: \mathbb{B}^2 \to \mathbb{R}^3$, having prescribed boundary values on $\partial\mathbb{B}^2$, whose mapping area, the integral of the Jacobian of f, was as small as possible. Indeed, it was for solving this formulation of Plateau's problem **[DJ]** that J. Douglas received one of the first two Fields Medals in 1936;[7] E. Bombieri received one of the two Fields Medals in 1974 for his work on higher-dimensional minimal surfaces and number theory.

Here are three difficulties with the mapping approach in general.

(1) *The filigree problem.* A flat isometric imbedding of the unit disk into space is a mapping of least area spanning its circle boundary. One can, however, pull out long thin tubes from the image disk having very small area, and indeed can construct a sequence of mappings or sets which have

[6] E. R. Reifenberg facetiously suggested the value of one of his basic constants to be 2^{-2000N}.

[7] As pretty a result as it is, it was accomplished basically by a trick. Area minimization (which is hard) was replaced by minimization of Dirichlet's integral (which is easier) among varying boundary values parametrizing the same image. The English edition of the treatise devoted to this type of minimal surface is J. C. C. Nitsche's *Lectures on Minimal Surfaces* **[NJ]**.

more and more tubes whose total area decreases to that of the flat disk; see Figure 1.3.4 in **[M1]**. Limit points of surfaces belonging to such a sequence can fill space.

(2) *The parametrization problem.* Suppose f is a flat isometric imbedding of the unit disk into space carrying $(0, 0)$ to $(0, 0, 0)$. It is easy to compose f with diffeomorphisms of the disk, which are the identity on the disk boundary, to obtain a sequence of mappings for which the image of every nonboundary point converges to the origin in space. The limit map is an unreasonable candidate for a least-area mapping.

(3) *Problems with variable domains.* It is not difficult to construct a simple closed unknotted curve of finite length which is a smooth embedding except at one complicated point, such that the infimum of areas of surfaces of genus k decreases as $k \to \infty$. This means that if one wants to find a surface of absolutely least area (which does exist) then it is going to be infinitely complicated topologically.

What can measure theory do about these difficulties? The general spirit of modern geometric measure theory is to make measures out of surfaces. If S is a surface in space, then the associated "variation measure" $\|S\|$ is that measure *on all of space* which assigns to an open subset A of space the surface area of $S \cap A$. The usual weak compactness theorem for measures guarantees that the sequence of measures associated to an area-minimizing sequence of surfaces spanning a given boundary will have a subsequence which converges to a limit measure; filigree just disappears in the limit and there are no parametrizations. The real work then is to recover the geometric content of the limit measure. If, for example, the S's in the area-minimizing sequence were all oriented submanifolds spanning a closed boundary curve C of finite length, then the limit measure would be associated with a two-dimensional real analytic minimal submanifold which would be a manifold with boundary in case C is smooth; this is one of the nicest theorems in geometric measure theory **[FW, HS]**. In higher codimensions the surface would be a branched minimal immersion **[CS]**; boundary regularity is not known.

Where does the name geometric measure theory come from? The name "geometric measure theory" came into existence as the title of H. Federer's treatise **[F1]** written in 1969.[8] The field has a "classical period" from about 1900 to 1960 and a "modern period" from about 1960 to the present.

How big a field of mathematics is geometric measure theory? I would guess that the number of published research pages in geometric measure theory in the past thirty years is many tens of thousands. I would guess that the number of living mathematicians who have published papers in the field numbers perhaps two hundred.

What general books have been written about least-area problems in a

[8] Federer's treatise would have been called geometric integration theory had not H. Whitney already published a book with that title. *Geometric Measure Theory* unfortunately is no longer in print despite my protests.

geometric measure theory setting? The treatise in the subject (up to 1969) is Federer's book above; many people find it difficult going. An introduction to the field is F. Morgan's *Geometric Measure Theory. A Beginner's Guide* **[M1]**. We also call attention to H. B. Lawson's *Lectures on Minimal Submanifolds* **[LB]**, L. Simon's *Lecture Notes on Geometric Measure Theory* **[S1]** and A. T. Fomenko's two volume work *Plateau's Problem* **[F2]**. The proceedings **[AA2]** of the 1984 AMS summer research institute devoted to geometric measure theory and minimal surfaces effectively surveys the field as of that date; it also contains a list of open problems and gives suggested terminology.

What is classical geometric measure theory? The first sixty years of the twentieth century was devoted to development of a theory of measure and integration for k-dimensional sets in n-dimensional spaces, especially when these sets might have essential singularities. Many of these developments are indispensable in modern geometric analysis. I feel that k rectifiable subsets are now well established as natural domains of integration using Hausdorff's k-dimensional measure $\mathscr{H}^k$. Essentially anything which is true for Lebesgue integration using Lebesgue's measure $\mathscr{L}^k$ on Euclidean spaces $\mathbb{R}^k$ remains true for Lebesgue integration using Hausdorff's measure on rectifiable sets. The Hausdorff area formula **[FH1,** 3.2.5] (a general change of variables formula) and Federer's coarea formula **[FH1,** 3.2.12, 3.2.22] (a curvilinear form of Fubini's theorem) imply that within the category of rectifiable sets and Lipschitz maps, everything which one could reasonably hope to be true is true provided one adds phases "$\mathscr{H}^k$ approximately", "$\mathscr{H}^k$ almost everywhere", etc. in appropriate places. Other major successes of classical geometric measure theory are the structure theorem for sets of finite Hausdorff measure **[FH1,** 3.3.13] due to A. S. Besicovitch and Federer, and the general Gauss-Green theorem **[FH1,** 4.5.6] proved by E. De Giorgi and Federer.

What are rectifiable sets, the structure theorem, and the general Gauss-Green theorem? A set S in $\mathbb{R}^n$ is k rectifiable[9] provided for each $\epsilon > 0$ there is a continuously differentiable submanifold $\mathscr{M}_\epsilon$ such that $\mathscr{H}^k\,[(\mathscr{M}_\epsilon \sim S) \cup (S \sim \mathscr{M}_\epsilon)] < \epsilon$. The structure theorem for sets of finite Hausdorff measure asserts that any set of finite Hausdorff measure is the almost unique disjoint union of a rectifiable set (which is effectively a smooth submanifold) together with a purely unrectifiable set which projects to measure zero under almost all orthogonal projections $\mathbb{R}^n \to \mathbb{R}^k$, and thus is effectively invisible to k-dimensional vision. Such purely unrectifiable sets of positive measure do exist; most examples are Cantor set type constructions. The general Gauss-Green theorem says that if the integral of the divergence of a Lipschitz vector field over a set is expressible by integration of the vector field at all, then that integration is necessarily over an oriented rectifiable set (called the "reduced boundary") with respect to Hausdorff measure.

What are the main successes of modern geometric measure theory? Among

[9] Federer calls these sets $(\mathscr{H}^k, k)$ *rectifiable* and $\mathscr{H}^k$ measurable **[FH1,** 3.2.14].

the principal achievements of modern geometric measure theory is creation of spaces of k-dimensional surfaces in n-dimensional spaces having sufficient compactness properties to permit use of the direct method in the calculus of variations (convergent subsequences of minimizing sequences); these theorems have given existence of surfaces of least area spanning boundaries or representing homology classes, etc. in a number of different contexts. These existence theorems have been either accompanied by or followed by a number of strong regularity and singularity classification theorems for minimizing surfaces. These "almost everywhere regularity" theorems **[GE, R2, A3]** came before and led to corresponding regularity theorems for nonlinear elliptic systems of partial differential equations, such as would be the Euler-Lagrange partial differential equations for the variational problems. My regularity theory for $(\mathbb{F}, \epsilon, \delta)$ minimal sets **[A4]** came before and led to the regularity theory of quasi-minimizing functions. Other successes are a variational calculus in the large for minimal surfaces, studies of other surface energy functions, the crystalline variational calculus, the theory of multifunctions, and the theory of calibrations.

Can geometric measure theory help model natural phenomena? A physical system will be in equilibrium when its free energy is not decreased by small variations. When surface energy is an important component of a system one wishes to model there is a good chance that geometric measure theory will be an important tool, perhaps indispensable; see, for example, "A mathematical contribution to Gibbs's analysis of fluid phases in equilibrium" **[AG]** by M. E. Gurtin and me. Indeed, a significant part of geometric measure theory has been developed with such uses in mind. The surfaces of geometric measure theory have provided the first realistic mathematical models for such phenomena as soap films on wire frames, soap bubble clusters, and grain boundaries in metals; see, for example, the expository articles **[A5]** and **[AT]**. J. Taylor showed mathematically in **[T1]** that the observed local structure of soap films and soap bubble clusters was a consequence of the assumption of area minimization. J. Cahn and Taylor predicted a cusp-type singularity in the surface of a metal which was later recognized in photographs **[CT]**; such geometry had previously been thought to be a dynamic phenomenon rather than an equilibrium configuration. J. Taylor's crystalline variational calculus has produced important new insights into the faceted surfaces which minimize crystalline surface energies.

The evolution of physical systems driven in part by surface free energy can lead to interface geometries whose topology and combinatorial structure changes with time. The evolution of grain boundaries in an annealing metal is such a phenomenon. K. Brakke's pioneering book *The Motion of a Surface by Its Mean Curvature* **[B1]** provides the first realistic mathematical model for such evolution.[10] It helped correct several mistakes in the

[10] Other curvature-driven motions have been and are being studied mathematically at the present time. Some of these are discussed, with references, in **[W2]**.

metallurgy literature. The paper "Mathematical existence of crystal growth with Gibbs-Thomson curvature effects" by L. Wang and me seems to give the first mathematical existence theorem for a model of a nonisotropic crystal freezing from its melt limited by the rate at which latent heat can diffuse away; for isotropic surface energies, note the contribution of S. Luckhaus [**LS**].

Techniques of geometric measure theory led to the first proof of the positive mass conjecture in general relativity [**SY**] and have led to new theorems for the analysis of liquid crystal geometries [**SU, HKL, ABL, AL**].

What are the main limitations of geometric measure theory? The first main limitation is that the subject is difficult and complicated with a number of long (and in some cases unpublished, including some of my own) papers. Also some of the most intriguing problems may be so difficult as to not be accessible to the present generation of mathematicians (such as classification of singularities). On the other hand, interesting problems come in various levels of difficulty, and the number of beautiful and important problems exceeds the number of mathematicians working on them.

How did modern geometric measure theory come into being? The year 1960 is frequently taken as a turning point in geometric measure theory and the geometric calculus of variations because of three seminal contributions:

First was the paper "Normal and integral currents" [**FF**] by H. Federer and W. H. Fleming; this paper was awarded one of the Society's 1987 Steele Prizes. The paper begins with the statement:

> "Long has been the search for a satisfactory analytic and topological formulation of the concept " k dimensional domain of integration in euclidean n-space". Such a notion must partake of the smoothness of differentiable manifold and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains must have certain compactness properties. All these requirements are met by the *integral currents* studied in this paper."

The best introductory reference to the mathematics which grew from this start is Morgan's book mentioned above.

A second contribution was work of E. De Giorgi which included, in particular, an almost everywhere regularity theorem for area-minimizing oriented hypersurfaces. Even though this theory is formally a mathematical subset of the theory of integral currents, it has its own structure and, for the study of area-minimizing hypersurfaces, is more readily accessible. The best introductory reference is the book *Minimal Surfaces and Functions of Bounded Variation* [**GE**] by E. Giusti.

The third contribution in 1960 was the paper "Solution of the Plateau problem for m dimensional surfaces of varying topological type" [**R1**], by E. R. Reifenberg which gave the first almost everywhere manifold structure for area-minimizing surfaces of general codimension.

Reifenberg was killed in a mountaineering accident in the Dolomites in the mid 1960s. De Giorgi, Federer, and Fleming are still alive.

What are currents, why are they called that, and where did they come from? Currents by definition are continuous linear functionals of vector spaces of smooth differential forms. They are called "currents" by analogy with the currents of electrical networks.[11] Since the cornerstone of modern geometric measure theory was the theory of currents, it seems useful to place them in historical perspective. Suppose $\mathscr{M} = \mathscr{M}^{m+n}$ is a smooth oriented submanifold of $\mathbb{R}^N$ without boundary and that $\Sigma = \Sigma^n$ is a surface in $\mathscr{M}$ representing an n-dimensional integral homology class $\tau \in \mathbb{H}_n(\mathscr{M}; \mathbf{Z})$.

Our manifold $\mathscr{M}$ supports differential forms of various dimensions. A pairing between forms which is of special interest to us is

$$(\text{differential } m\text{-forms})_c \times (\text{differential } n\text{-forms}) \to \mathbb{R},$$

$$(\sigma, \omega) \mapsto \int_{\mathscr{M}} \sigma \wedge \omega.$$

On the cohomology level this pairing induces a mapping

$$\mathbb{H}_c^m(\mathscr{M}; \mathbb{R}) \otimes \mathbb{H}^n(\mathscr{M}; \mathbb{R}) \to \mathbb{R}.$$

Poincaré duality is the statement that this pairing identifies $\mathbb{H}_c^m(\mathscr{M}; \mathbb{R})$ as the dual of $\mathbb{H}^n(\mathscr{M}; \mathbb{R})$. In particular, any linear function on $\mathbb{H}^n(\mathscr{M}; \mathbb{R})$ corresponds to some closed differential m-form σ.

Our closed surface Σ also induces a linear function

$$\mathbb{H}^n(\mathscr{M}; \mathbb{R}) \to \mathbb{R}, \qquad \omega \mapsto \int_\Sigma \omega.$$

This leads to several interesting problems.

First Problem. Find a closed differential m-form representing the n-dimensional closed surface Σ. The basic difficulty here is that the normal bundle of Σ need not be trivial. It was problems such as this which were part of the motivation for H. Whitney and H. Hopf and others to develop the theory of fiber bundles in the period between 1935 and 1940.

Second Problem. Find a closed n-dimensional surface Σ representing the closed differential m-form σ. One of the difficulties here is that it is not always possible for Σ to be a manifold. This situation was one of the motivations for G. de Rham to introduce his theory of currents **[RG]** in 1955—by definition, a current is indeed a continuous linear functional on

[11] Kirchhoff's laws for simple electrical networks have the following interpretation. Make a real 1-chain out of the network by orienting each simple wire by the direction of electrical current flow and giving it density equal to the current flowing. Then integration of differential 1-forms makes the network 1-chain into a mathematical current. Kirchhoff's first law that the electrical current flowing into each node must algebraically sum to zero is equivalent to the statement that the mathematical 1-chain is a 1-cycle. Now consider the 1-cochain which assigns to each simple wire the voltage drop along it. Kirchhoff's second law that the sum of voltage drops around a closed loop be zero is equivalent to the statement that this cochain be a cocycle.

differential forms. It was also one of R. Thom's motivations for developing his cobordism theory.

What is a integral current? A *k-dimensional rectifiable current* $T = \mathbf{t}(S, \theta, \xi)$ in $\mathbb{R}^n$ consists of

- an underlying k rectifiable set S,
- a positive integer-valued density function $\theta : S \to \mathbf{Z}^+$;
- a unit simple k vector-valued orientation function $\xi : S \to \bigwedge_k \mathbb{R}^n$.

T is called a *complex current* provided the real linear space associated with each $\xi(x)$ is a complex linear subspace. T is called a *real rectifiable current* in case the density function θ above is permitted to take positive real number values as well as integer values.

The *size* $\mathbb{S}(T)$ of T is, by definition, Hausdorff's k-dimensional measure $\mathscr{H}^k$ of the underlying set,

$$\mathbb{S}(T) = \mathscr{H}^k(S) = \int_S 1\, d\mathscr{H}^k.$$

The *mass* $\mathbb{M}(T)$ of T is the area of the underlying set, weighted with its density; i.e.,

$$\mathbb{M}(T) = \int_S \theta\, d\mathscr{H}^k.$$

As a k-dimensional "de Rham current," our rectifiable k-current T maps a smooth differential k-form φ to the number

$$T(\varphi) = \int_{x \in S} \langle \xi(x)\, ,\, \varphi(x) \rangle\, \theta(x)\, d\mathscr{H}^k x.$$

The boundary of T is the general current

$$\partial T(\omega) \equiv T(d\omega),$$

so that in geometrically reasonable cases Stokes's theorem is a definition. Our rectifiable k-current T is called an *integral current* provided ∂T is a rectifiable $(k-1)$-current.

The integral currents lying on a submanifold of $\mathbb{R}^n$ form a chain complex analogous to the singular chains with integer coefficients of algebraic topology and with similar homological properties. Spaces of integral currents map naturally under Lipschitz mappings of their ambient spaces and also by Lipschitz multifunction mappings of the their ambient spaces as discussed below.

There are a number of reasons for identifying oriented surfaces with currents. Among them are the following.

- The current structure automatically identifies different parametrizations of the same underlying surface and also identifies surfaces differing on sets of measure zero (which is what one normally wants to do in the geometric calculus of variations).
- The current structure provides a well-defined notion of surface boundary even in the presence of considerable geometric complexity; some integral

currents are carried by rectifiable sets which are everywhere locally infinitely connected.

- By identifying surfaces as currents various spaces of oriented surfaces acquire a natural weak topology with respect to which very strong compactness theorems hold.

What is a parametric integrand and what is its integral? In a traditional multiple integral calculus of variations, an integrand is a function $F(x, y, p)$ defined for $x \in \Omega \subset \mathbb{R}^k$, $y \in \mathbb{R}^n$, and $p \in \operatorname{Hom}(\mathbb{R}^k, \mathbb{R}^n)$. To a function $f: \Omega \to \mathbb{R}^n$ is then associated the integral

$$\mathbb{F}(f) = \int_\Omega F(x, f(x), Df(x))\, d\mathscr{L}^k x.$$

In case $\mathbb{F}(f \circ g) = \mathbb{F}(f)$ for all diffeomorphisms $g: \Omega \to \Omega$, then C. B. Morrey, Jr. termed the integrand F "parametric"; otherwise F is "nonparametric"; see **[MC]**. The k-dimensional parametric area integrand assigns to (x, y, p) the Jacobian of p; its integral thus equals the area of $f(\Omega)$ computed with multiplicities, and its minimizers are minimal surfaces. The nonparametric Dirichlet's integrand assigns to (x, y, p) the Euclidean norm squared $|p|^2$ of p; its minimizers are harmonic functions. To complicate matters there is a nonparametric area integrand whose integral with respect to f equals the area of the graph of f.

If $F(x, y, p)$ is a parametric integrand then there is a function $G(z, \xi)$ defined for $z \in \mathbb{R}^n$ and ξ belonging to the Grassmann manifold $\mathbf{G}_0(n, k)$ of oriented k-dimensional linear subspaces of $\mathbb{R}^n$ such that, in general,

$$\int_\Omega F(x, f(x), Df(x))\, d\mathscr{L}^k x = \int_{z\in S} G(z, \operatorname{Tan}(S, z))\, N(f, z)\, d\mathscr{H}^k z;$$

here $S = f(\Omega)$ is the image of f, $\operatorname{Tan}(S, z)$ is the oriented tangent plane to S at z, and $N(f, z)$ is the multiplicity with which f assumes the value z. Such a G exists if and only if F is parametric. In geometric measure theory one usually forgets about F and f and thinks of a parametric integrand as a function $G: \mathbb{R}^n \times \mathbf{G}_0(n, k) \to \mathbb{R}$ which can be integrated over oriented rectifiable sets with densities. Since $\mathbf{G}_0(n, k)$ can be naturally identified with the unit simple k-vectors in the Grassmann vector space $\bigwedge_k \mathbb{R}^n$ there is always a function $\Phi: \mathbb{R}^n \times \bigwedge_k \mathbb{R}^n \to \mathbb{R}$ which is positively homogeneous of degree one whose restriction to unit simple vectors coincides with G. The parametric integrand associated to a differential k-form $\omega: \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ assigns to (z, ξ) the number $\langle \xi, \omega(z) \rangle$. To each such parametric integrand Φ and each integral current $T = \mathbf{t}(S, \theta, \xi)$ is associated the *integral of* Φ *over* T

$$\int_T \Phi = \int_{z\in S} \Phi(z, \xi(z))\, \theta(z)\, d\mathscr{H}^k z.$$

What is an elliptic integrand and why are they important? A parametric integrand $\Phi: \mathbb{R}^n \times \bigwedge_k \mathbb{R}^n \to \mathbb{R}$ is called "elliptic" (for integral currents) provided it is continuous and there is a positive continuous function $\epsilon: \mathbb{R}^n \to \mathbb{R}$

such that

$$\int_T \Phi_{z_0} - \int_{T_0} \Phi_{z_0} \geq \epsilon(z_0)\left(\mathbb{M}(T) - \mathbb{M}(T_0)\right)$$

whenever $z_0 \in \mathbb{R}^n$, $T_0 = \mathbf{t}(D_0, \omega_0, \xi_0)$ is an oriented flat k-disk in $\mathbb{R}^n$ with constant density ω_0 and constant orientation ξ_0, and T is any integral current with $\partial T = \partial T_0$; here $\Phi_{z_0}(z, \xi) = \Phi(z_0, \xi)$.

I introduced the notion of ellipticity in **[A3]** as the natural condition on an integrand in order that smooth regularity hold for minimizers; see **[A3, A4, ASS]**. It also seemed important to me at that time that geometric measure theory not be excessively focussed on least area problems. If a smooth minimizer is written locally as the graph of a function, then the Euler-Lagrange partial differential equations this function must satisfy are a strongly elliptic system of partial differential equations; this was my reason for calling such integrands elliptic.

The positivity and uniform convexity (in the ξ variables) of a parametric integrand are sufficient in order that it be elliptic for integral currents. The notion of ellipticity for other types of surfaces is defined in a similar way. Any integrand which is sufficiently close to the area integrand in the C^2 topology is elliptic for any type of surface. A general characterization of ellipticity does not yet exist.

What are some of the main theorems about integral currents? Suppose $\mathscr{M} = \mathscr{M}^{m+n}$ is a smooth compact submanifold of $\mathbb{R}^N$ without boundary and that τ is an n-dimensional integral homology class in $\mathbb{H}_n(\mathscr{M}; \mathbf{Z})$.

THEOREM (Compactness and semicontinuity). *Suppose that $T_1, T_2, T_3, \ldots$ are integral currents in $\mathscr{M}$ such that $\sup_i \mathbb{M}(T_i) + \mathbb{M}(\partial T_i) < \infty$. Then there is a subsequence $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$ and an integral current T such that $T_{i(j)} \to T$ weakly (equivalently in the integral flat norm topology) as $j \to \infty$. Furthermore, mass and size are lower semicontinuous under this convergence, as is the integral of any elliptic integrand.*

THEOREM (Homology groups) **[FF]**. *The homology groups of the chain complex of integral currents on $\mathscr{M}$ are naturally isomorphic with the singular homology groups of $\mathscr{M}$ with integer coefficients. Furthermore, integral homology classes are preserved under convergence of integral cycles in the integral flat norm.*

THEOREM (Homotopy groups) **[A1]**. *The ith homotopy group of the group of k-dimensional integral cycles on $\mathscr{M}$ is naturally isomorphic with the $(i+k)$th integral homology group of $\mathscr{M}$; i.e.,*

$$\pi_i(\mathscr{Z}_k(\mathscr{M}); 0) \cong \mathbb{H}_{i+k}(\mathscr{M}; \mathbf{Z}).$$

These theorems help produce minimal surfaces by direct methods and by methods in the large. Perhaps the centerpiece theorem of integral current theory is the following.

THEOREM (Optimal representatives of integral homology classes). (1) *The integral homology class τ can be represented by an integral current T of least mass or minimizing the integral of any continuous elliptic integrand* **[FF]**.

(2) *Any mass-minimizing integral current T representing τ is a smooth minimal submanifold of $\mathscr{M}$ except for a possible singular set of codimension at least two* **[A6]**; *the singular set will have codimension at least seven in case $m = 1$* **[FH2]**.

(3) *The singular set of any integral hypersurface minimizing the integral of a smooth elliptic integrand has codimension exceeding two* **[ASS]**.

(4) *Each two-dimensional mass-minimizing integral current is a classical branched minimal immersion* **[CS]**.

(5) *Suppose $\mathscr{M}$ is simply connected and f is function from a compact connected oriented n-dimensional manifold $\mathscr{N}$ without boundary into $\mathscr{M}$ whose associated homology class is τ. In case $n \geq 3$ and some mass-minimizing integral current T representing τ can be finitely triangulated* (*as is known to be the case for $n = 3, 4, 5, 6, 7$ in case $m = 1$*), *then there is a Lipschitz function g homotopic to f and of least mapping area among all such functions* **[W1]** (*the mapping area of g is the integral over $\mathscr{N}$ of the Jacobian of g*).

Since holomorphic varieties (including those with singularities) in Kähler manifolds minimize mass in their homology classes (see below), mass-minimizing surfaces can have complicated singularities and the codimension two estimate in (2) above is the best possible in general codimensions. The codimension seven estimate for mass-minimizing hypersurfaces is also optimal since the central cone over $\mathbf{S}^3 \times \mathbf{S}^3$ in $\mathbb{R}^8$ is mass minimizing.

Thom's cobordism theory shows that not all τ's can be represented by a map f as in (5) above from any domain manifold. Should a mass-minimizing integral current T be supported by a real analytic set, its singularities could be "resolved" thereby giving such a mapping **[MA]**. Cobordism thus implies an obstruction not only to regularity but also to real analyticity of singular sets.

Results like that in (5) above effectively reduce classical least area mapping problems in higher dimensions to questions about the singular structure of mass-minimizing integral currents.

What is the integral flat norm and why is it called that? The integral flat norm of a k-dimensional integral current T is the infimum of the numbers $\mathbb{M}(T - \partial Q) + \mathbb{M}(Q)$ corresponding to all $(k+1)$-dimensional integral currents Q. Such a minimizing Q is often of use in geometric constructions with currents. The term "flat norm" was introduced by H. Whitney in **[WH]** along with a "sharp" norm with musical notations.[12]

What are tangent cones and why are they important. Suppose S is a k-dimensional minimal surface in $\mathbb{R}^n$ containing the origin as an interior point.

[12]Whitney majored in both music and mathematics as an undergraduate at Yale.

Let $A(r)$ denote the area of S inside the unit r ball centered at the origin. The important *monotonicity formula* for minimal surfaces tells us that the function $r \to r^{-k}A(r)$ is nondecreasing. Now let S_r be the dilation of S by factor r. Compactness theorems imply the existence of a sequence $r(1), r(2), r(3), \ldots$ of radii converging to 0 and a minimal cone S_* such that $S_{r(j)} \to S_*$ as $j \to \infty$. S_* approximates S near the origin but is simpler because it is a cone. One can then take a tangent cone to the tangent cone, other than at the origin, to get an even simpler approximation, etc. Such downward induction was a key step in **[FH2]**. Many examples of area-minimizing tangent cones and partial classification of them is given by Lawlor in **[LG]**.

For crystalline surface energies in space J. Taylor gave a complete classification of tangent cones in **[T2]**.

What is the uniqueness of tangent cones problem? It is a problem of long standing to settle the question whether or not tangent cones to minimal surfaces are unique. Could one get a different limiting tangent cone if a different sequence of radii converging to zero above were chosen? If the cone S_* is singular only at the origin, then it is the unique cone according to L. Simon in "Asymptotics for a class of non-linear evolution equations with applications to geometric problems" **[S2]**; another criterion is given in **[AA1]**.

Can mass-minimizing integral currents be approximated by submanifolds? How closely can one approximate a general integral k current T representing a given integral homology class τ in an n-dimensional Riemannian manifold $\mathscr{M}$ by a smooth k-dimensional submanifold? This is a basic question treated in **[AB]** using Thom's criterion in the context of homotopy classes of mappings from $\mathscr{M}$ (less a skeleton) to the Thom complex $T(\gamma^{n-k})$. One of the principal results is the following.

THEOREM (Optimal smooth approximation of integral cycles) **[AB]**. *Suppose $\epsilon > 0$ and T is any integral cycle representing an homology class τ. Then there is a smooth triangulation of $\mathscr{M}$ and an oriented k-dimensional submanifold Σ of $\mathscr{M}$ with its $k-5$ skeleton removed with the following properties.*

(1) *The k area of Σ does not exceed the the mass of T by more than ϵ.*

(2) *The current $[\![\Sigma]\!]$ is an integral cycle homologous to T. Furthermore, this homology is representable by an integral $(k+1)$-dimensional current Q in $\mathscr{M}$ (i.e. $\partial Q = [\![\Sigma]\!] - T$) having mass less than ϵ.*

(3) *If the homology class τ is representable by some submanifold, then the approximating Σ above can be chosen to be a submanifold.*

In case the codimension $n-k$ equals 1 or 2, then τ is carried by some submanifold and the approximating Σ's of the theorem above can be chosen to be free of all singularities. In general whenever τ is carried by a submanifold, approximating Σ's without singularities can be selected from any bordism class associated with τ.

Examples show that the codimension five estimate in the theorem above is the best possible general estimate.

Suppose in the theorem above that T is mass minimizing and τ is carried by some submanifold. Examples show that it is not always possible to require that the approximating submanifold Σ be close to the support of T, or close to the support of any mass-minimizing current representing τ.

As one consequence of these estimates we have the following.

COROLLARY. *Suppose either $m = 1, 2$ or $n = 2, 3, 4$. Then each mass-minimizing integral current T representing τ is the limit of a minimizing sequence of integral currents associated with n-dimensional submanifolds of $\mathscr{M}$ within any bordism class of such manifolds.*

What are multifunctions and what are they good for? A typical multifunction is defined on a domain Ω in $\mathbb{R}^m$ and takes values in the group of zero dimensional integral currents in $\mathbb{R}^n$, i.e. finite sets of points with integer multiplicities. Such functions are powerful and natural tools which for some problems in geometric measure theory are indispensable. I feel that the theory of integral and real rectifiable currents ultimately will be largely formulated in the language of multifunctions. Since flat chains mod ν (discussed below) are a quotient of integral currents, there is a corresponding multifunction theory for them.

The multiply sheeted "graph" of a multifunction can represent an oriented m-dimensional surface in $\mathbb{R}^{m+n}$. An example of a multi-function is

$$f(x) = \begin{cases} [\![x^{1/2}]\!] - [\![-x^{1/2}]\!] & \text{if } x > 0, \\ \\ 0 \quad \text{if } x \le 0 \end{cases}$$

defined for all real numbers x. The "graph" of f is the 1-current associated with the parabola $x = y^2$ suitably oriented in the plane; see [A9, ASu]. Formally, one would write this oriented parabola as the current $(1_{\mathbb{R}} \bowtie f)_\# \mathbb{E}^1$. General Lipschitz multifunctions induce natural chain mappings on chain complexes of integral currents. When one does approximate a complicated surface as the graph of a multifunction on a fixed simple domain one is led to applications of techniques of functional analysis in ways novel to essentially geometric problems.

One of the basic techniques in the analysis of mass-minimizing integral currents is approximation of such currents as graphs of multifunctions taking values in (the 0-current equivalent of) unordered Q tuples of points [A6, CS]. Such multifunctions nearly minimize an appropriately defined Dirichlet's integral. A priori estimates on Dirichlet's integral minimizing multifunctions then lead to estimates on mass minimizing integral currents. As an example, the function which maps the complex number z to the unordered pair of square roots of z^3 is Dirichlet's integral minimizing; its graph is the mass-minimizing complex curve $z^3 = w^2$ which is branched at the origin.

The general theory of multifunctions taking values in zero-dimensional real polyhedral chains is set forth in [A7]. A pivotal theorem there asserts that an arbitrary integral cycle can be approximated by the graph of a Lipschitz multifunction with strong a priori estimates on the Lipschitz constant, multiplicity, and the size of the set in the domain on which the approximation fails to be exact. This approximation theorem is strong enough to be the basis of a proof of the compactness theorem for integral currents mentioned above based on the compactness properties of equicontinuous families of functions. It also provided the first compactness theorem for size-bounded real rectifiable currents.

What are normal currents? A current is called (locally) *normal* provided that it and its boundary are representable by integration. Such currents were introduced, together with integral currents, in **[FF]**. The theory of normal currents is less developed than the theory of rectifiable currents. One principal paper in the field is "Real flat chains, cochains and variational problems" **[FH3]**. It contains a number of theorems including generalization of the classical theorems of de Rham about the duality between smooth singular chains and smooth differential forms.[13] A second principal paper in the field is "The second variation of normal currents" **[St]** by J. Steinke. It gives alternative proofs of the results based on S. Bochner's procedure for showing the vanishing of real homology groups in the presence of positive curvatures; it also gives a new curvature formula by analysis of the mass norm on minimizing normal currents.

What are integral varifolds? The homotopy information about currents above gives sufficient topological information to construct a variational calculus in the large for the area integrand. The, generally unstable, minimal surfaces one produces do not necessarily have the structure of currents because of possible cancellations. Varifold surfaces were introduced by me in "The theory of varifolds. A variational calculus in the large for the k dimensional area integrand" **[A2]** to make this theory work. I called the new objects "varifolds" having in mind that they were a measure-theoretic substitute for manifolds created for the variational calculus. A varifold V has a natural first variation distribution δV defined on initial velocity vector fields in the ambient space. Strong compactness and regularity theorems hold for integral varifolds with controlled first variation distributions according to W. K. Allard in papers "On the first variation of a varifold" **[AW1]** and "On the first variation of a varifold; Boundary behavior" **[AW1]**.

A k-dimensional *integral varifold* V can be expressed

$$V = \mathbf{v}(S, \theta, \tau)$$

corresponding to some k rectifiable set S together with a positive integer-valued density function θ and an unoriented tangent plane indicator τ. A

[13]The open problem suggested on p. 400, line 7 has been settled in the negative by a counterexample I found.

general k-dimensional varifold in $\mathscr{M}$ is a Radon measure over the Grassmann bundle over $\mathscr{M}$. For example, if $A \times B \subset \mathbb{R}^n \times \mathbf{G}(n, k)$, then

$$\mathbf{v}(S, \theta, \tau)(A \times B) = \int_{S \cap A \cap \{x : \tau(x) \in B\}} \theta \, d\mathscr{H}^k.$$

"Minimal varifolds" are called stationary. The following theorem represents the combined work of Allard, Almgren , S. Chang, J. Pitts, R. Schoen, and L. Simon.

THEOREM (Existence and regularity of stationary integral varifolds) **[PJ]**. *Each smooth compact Riemannian manifold $\mathscr{M}^{m+n}$ supports at least one n-dimensional stationary integral varifold V which is a regular minimal surface on an open dense set. If V is a hypersurface then the singularities in V are of codimension at least seven. If $n = 2$ then V is a branched minimal immersion.*

Every manifold thus supports at least one minimal surface in each dimension. It is not known if there are always infinitely many. One variant of this theorem for surfaces with boundary is the following.

THEOREM (Two minimal surfaces guarantees a third). *If a curve C in space bounds two distinct minimal surfaces which are locally (among nearby surfaces) of least area, then it bounds at least one more minimal surface which is regular in the complement of C.*

J. Pitts and J. H. Rubinstein have used the variational calculus in the large with symmetry constraints to produce large new classes of regular minimal submanifolds the 3-sphere and of 3-space **[PR]**.

What are flat chains mod ν? In the same way that integral currents are the measure-theoretic version of singular chains with integer coefficients, the "flat chains mod ν" are the measure-theoretic version of singular chains with coefficients in the integers modulo ν **[FH1, 4.2]**; they were introduced by W. P. Ziemer in **[ZW]** and W. H. Fleming in **[FH2]**. In order to build a variational calculus which would produce, for example, a Möbius band as an unoriented surface of least area, one is led to consider the quotient of the integral currents by two times the integral currents [more generally, by ν times the integral currents]. This gives the flat chains mod 2 [mod ν]. It is a new theorem of mine that this space is complete **[A12]**; e.g., one does not need to pass to a completion in order to solve variational problems, such as minimizing area.

One can use the language of currents and flat chains in order to give one form of my general isoperimetric inequality.

THEOREM (Optimal isoperimetric inequality) **[A8]**. *Corresponding to each $(m-1)$-dimensional integral cycle [resp. flat cycle mod ν] T in $\mathbb{R}^n$ there is an m-dimensional integral current [resp. flat chain mod ν] Q having T as boundary such that*

$$\mathbb{M}(Q) \leq \gamma(m)\mathbb{M}(T)\mathbb{S}(T)^{1/(m-1)} \leq \gamma(m)\mathbb{M}(T)^{m/(m-1)}$$

with equalities if and only if T is a standard round $(m-1)$-sphere (of some radius) and Q is the corresponding flat disk. The required equalities define $\gamma(m)$.

The left-hand inequality above also holds for real rectifiable currents and similar isoperimetric inequalities hold for stationary integral varifolds—the value of the *optimal* constant for varifolds is not known. A key step in the proof of the optimal isoperimetric inequality above is the following characterization of standard spheres.

THEOREM (Mean curvature characterization of standard spheres) **[A8]**. *Suppose V is an m-dimensional surface in $\mathbb{R}^n$ without boundary. If the mean curvature vectors of V do not exceed in length those of a standard round m-sphere of unit radius, then the m-area of V (actually of the extreme points of V) is not less than the m-area of the standard unit m-sphere. Furthermore, equality holds if and only if V is such a standard round unit m-sphere.*

What are $(\mathbb{M}, \epsilon, \delta)$ **minimal sets?** The $(\mathbb{M}, \epsilon, \delta)$ minimal sets are a special class of compact sets with boundaries which are not defined combinatorially. These sets were isolated by me in order to model soap films, soap bubble clusters, and combination bubble-films. Such minimal surfaces sometimes do not correspond to integral currents or flat chains mod ν; the central cone over the one-dimensional skeleton of a regular tetrahedron does not assume its boundary in any natural combinatorial way; see, however, **[LM]**. A surface Σ is called $(\mathbb{M}, \epsilon, \delta)$ minimal provided inside any ball U of radius ρ (not exceeding δ)

$$\mathscr{H}^2(\Sigma \cap U) \leq [1+\epsilon(\rho)]\, \mathscr{H}^2\, [\varphi(\Sigma \cap U)]\,,$$

for any Lipschitz map φ which moves points only inside U; this means that, inside U, the surface Σ comes within factor $[1+\epsilon(\rho)]$ of being area (size) minimizing. Similar "minimal surface forms" arise not only in the surface tension phenomena of liquids and thin films, such as soap bubbles, but also in grain boundaries in metals, in radiolarian skeletons, in close packing problems, in immiscible liquids in equilibrium, in sorting of embryonic tissues, in design, and in art **[A5]**. The existence and almost everywhere regularity theory for such surfaces in general dimensions and codimensions (and more general elliptic surface energy functions) appears in **[A4]**. For surfaces in space the singular structure is given by the following theorem.

THEOREM (Structure of soap bubble like minimal surfaces) **[T1]**. *Suppose $\epsilon(\rho) = C\rho^\alpha$ for some C and $0 < \alpha < 2$. Then any $(\mathbb{M}, \epsilon, \delta)$ minimal surface in $\mathbb{R}^3$ consists of of a finite number of Hölder continuously differentiable sheets of surface. These pieces of surface meet only along finitely many smooth arcs, where exactly three sheets must meet at equal angles of $120°$. Finally, these arcs can meet only at finitely many points, where six sheets* *of surface meet in triples along each of the four arcs meeting at the point. Since the sheets must meet at* 120° *angles, the arcs necessarily meet at the (approximately* 109°*) angles of a regular tetrahedron.*

What are crystalline minimal surfaces? Area is only one of the important surface energies. Crystalline energies are the most different and are also perhaps the most important. If one minimizes a parametric surface energy among all possible surfaces enclosing a given volume and the convex solution "Wulff shape" turns out to be a polyhedron then the surface energy is called "crystalline". Crystalline surface energies occur along interfaces of some crystalline solids. J. Taylor has discovered the following striking boundedness theorem for crystalline minimal surfaces.

THEOREM (Complexity bounds for crystalline minimal surfaces) **[T3]**. *The number of plane segments in a minimizing crystalline minimal surface in general position is bounded (explicitly and optimally) in terms of the number of boundary line segments, the surface genus, and the complexity of the Wulff shape.*

Using such bounds, J. Taylor developed an algorithm (based on dual graph analysis) for the computation of surfaces of least crystalline energies. Crystalline surface energies are also a stabilizing factor in crystal growth **[AW]**.

What are calibrations? A k-dimensional *calibration* of a Riemannian manifold $\mathscr{M}$ is a closed differential k-form ω such that $\langle \xi, \omega(x)\rangle \leq 1$ for each x in $\mathscr{M}$ and each unit simple k-vector ξ in $\bigwedge_k \operatorname{Tan}(\mathscr{M}, x)$. Such an ω *calibrates* an integral k-current $T = \mathbf{t}(S, \theta, \xi)$ provided $\langle \xi(x), \omega(x)\rangle = 1$ for $\|T\|$ almost every x. If $T_* = \mathbf{t}(S_*, \theta_*, \xi_*)$ is any other k-current which is homologous with T, i.e. $T - T_* = \partial Q$ for some Q, then

$$\begin{aligned} \mathbb{M}(T) &= \int_S \theta \, d\mathscr{H}^k = \int_S \langle \xi, \omega\rangle \, \theta \, d\mathscr{H}^k = \int_T \omega \\ &= \int_{T_*} \omega = \int_{S_*} \langle \xi_*, \omega\rangle \, \theta_* \, d\mathscr{H}^k \leq \int_{S_*} \theta_* \, d\mathscr{H}^k = \mathbb{M}(T_*) \end{aligned}$$

since

$$0 = \int_Q d\omega = \int_T \omega - \int_{T_*} \omega$$

by Stokes's theorem. Hence T minimizes mass in its homology class. Calibrations are thus a way of proving that a given surface is absolutely area minimizing. The general theory of calibrations was instituted in the paper "Calibrated geometries" **[HL]**, by R. Harvey and H. B. Lawson, Jr. as an extension of the fact, based on Wirtinger's inequality, that a constant times powers of the Kähler form calibrate holomorphic varieties in Kähler manifolds **[FH1, 5.4.19]**. The present rich theory is set forth in two review papers **[M2, M3]** by F. Morgan. A striking variant of calibration theory has been developed by G. Lawlor in "A sufficient criterion for a cone to be area-minimizing" [**LG**]. A thoroughly unlikely adaptation of calibration theory is used by Lawlor and Morgan [**LM**] to show that the central cone over the 1-skeleton of the regular tetrahedron is area minimizing as an interface between immiscible liquids. Note also T. Murdoch's twisted calibrations introduced in [**MT**].

If ζ is a k-vector in the Grassmann vector space $\bigwedge_k \mathbb{R}^n$ then the function which sends $\xi \in \bigwedge_k \mathbb{R}^n$ to $\eta(\xi) = \zeta \bullet \xi$ is a k covector. In case $\zeta \bullet \xi \leq 1$ for each unit simple k-vector ξ then the differential k-form $x \to \omega(x) = \eta$ calibrates $\mathbb{R}^n$. The *facet* of the Grassmann manifold of unit simple k-vectors determined by ζ consists of all such unit simple vectors ξ for which $\eta(\xi) = 1$. The facets determined by powers of the Kähler form turn out to be the unit simple complex vectors. One program of research in calibration theory has been been to find large facets and then attempt to construct integral currents with such tangent planes (thereby producing area minimizing surfaces); see [**A10**].

What are stratified minimal surfaces, stratified volumes, and spectral constraints? In the paper of Reifenberg mentioned above [**R1**], a surface was a compact set spanning a Čech homology class with area equal to spherical Hausdorff measure. Reifenberg obtained the first regularity results for area-minimizing surfaces in general dimensions and codimensions [**R2**] in 1964 by showing in $\mathbb{R}^n$ that his surfaces were almost everywhere smooth minimal submanifolds.

The Russian mathematician A. T. Fomenko[14] introduced interesting new constraints for area minimization called spectral homology and cohomology as discussed in his book *Variational Principles in Topology. Multidimensional Minimal Surface Theory* [**FA2**]; some caution is needed (see my review in the Bulletin of the Society, January 1992). Roughly speaking, what is required for such a constraint is preservation of nontriviality under convergence of compact sets in the Hausdorff distance topology. Since the smallest set carrying the nontriviality can be of lower dimension (the nontriviality of the normal bundle of a Möbius band in $\mathbb{R}^3$, for example, is carried by a curve) one is led to construct strata of minimal surfaces, each minimizing area in its own dimension subject to the condition that all higher dimensional pieces already minimize their area. Fomenko proves the existence and almost everywhere regularity of each stratum by an extension of Reifenberg's arguments; regularity alternatively follows from my paper [**A4**].

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[14]Fomenko is also an accomplished artist. Some of his work has been published by the American Mathematical Society [**F3**].

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